21-228 Discrete Mathematics Exam 1

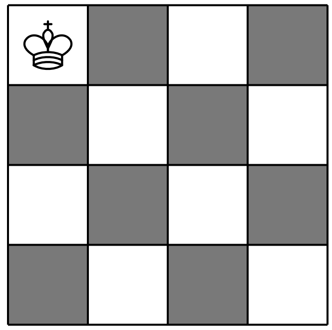
February 26, 2021

Problem	**Score**
1	
2	
3	
4	
5	
Total	

Name: ______________________________

This 50-minute exam is open-notes, in the sense that you may use anything you have written yourself. No calculators are permitted. Please write your answers in the space provided, and indicate clearly on the front of a page if you use the back of that page for additional space. Every numerical answer must be justified with an explanation. You may use any theorem which was stated in class without reproving it. Each problem is worth 10 points.

1. How many ways are there for a chess king to move from the top left square to the bottom right square of a 4×4 chessboard, if every move must be either to the right, down, or diagonally down and right?



2. Two identical decks of 52 cards are independently and separately shuffled, and placed side-by-side with all cards face down. Then, the top card from each deck is turned over at the same time, and then the next card from each deck is turned over at the same time, and so on, until the last card from each deck is turned over at the same time.

   What is the probability that there is at least one time at which the same card is turned over from both decks at the same time? Since calculators are not permitted, please provide an answer which is within 0.1 of the exact answer, in terms of any famous mathematical constants or mathematical symbols, except you may not use ellipses ("$\cdots$"), summation notation ("Σ"), or product notation ("Π"). Make sure to justify why your answer is within 0.1 of the exact answer.

3. Prove that no matter how 7 "$\times$" symbols are placed in the centers of different blank spaces in a 3×3 Tic-Tac-Toe board, you can always find 4 of the symbols which happen to be located at the corners of a perfect rectangle.

4. Simplify, and express this without using ellipses, summation notation, or product notation. You may use factorials. Note that this takes all binomial coefficients up to $\binom{n}{n}$, after which it takes fewer and fewer binomial coefficients in each subsequent row of Pascal's triangle.

$$\begin{aligned}
&\binom{0}{0} \\
&+\binom{1}{0}+\binom{1}{1} \\
&+\binom{2}{0}+\binom{2}{1}+\binom{2}{2} \\
&+\cdots \\
&+\binom{n-1}{0}+\binom{n-1}{1}+\cdots+\binom{n-1}{n-1} \\
&+\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n} \\
&+\binom{n+1}{1}+\binom{n+1}{2}+\cdots+\binom{n+1}{n} \\
&+\binom{n+2}{2}+\binom{n+2}{3}+\cdots+\binom{n+2}{n} \\
&+\cdots \\
&+\binom{2n-1}{n-1}+\binom{2n-1}{n} \\
&+\binom{2n}{n}
\end{aligned}$$

5. How many rectangles (with edges along grid lines) are there in an 8×8 chessboard, which contain the same number of black and white squares? One is indicated. The whole chessboard counts as a valid rectangle too. Rectangles must have nonzero area. Since calculators are not permitted, you may express your answer as a simple mathematical expression, as long as you do not use ellipses, summation notation, or product notation.