

15. 

(Just do it)

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CMU Putnam Seminar, Fall 2020

1 Classical results

Helly. Let C_1, C_2, \dots, C_n be a collection of convex subsets of \mathbb{R}^d , with the property that every $d+1$ of them have nonempty intersection. Then the whole collection has nonempty intersection.

Brouwer's Fixed Point Theorem. Every continuous function from a closed Euclidean ball to itself has a fixed point.

2 Problems

- (1913 entrance exam to Carnegie Institute of Technology, Math.) A spherical triangle has angles of 70° , 90° , and 100° , and the underlying sphere has radius 10. What is the area of the spherical triangle?
- (1913 entrance exam to CIT, English.) What is the feminine form of the noun "duck"?
- Show that there are exactly $\binom{n-k+1}{k}$ subsets of $\{1, 2, \dots, n\}$ with k elements and not containing both i and $i+1$ for any i .
- Does the series $\sum_{n=2}^{\infty} \frac{1}{\log(n!)}$ converge?
- A finite set of circles divides the plane into regions. Show that we can color the plane with two colors so that no two adjacent regions (with a common arc of non-zero length forming part of each region's boundary) have the same color.
- Let S be a finite collection of closed intervals on the real line such that any two have a point in common. Prove that the intersection of all the intervals is non-empty.
- Let S be a set and P the set of all subsets of S . Let $f : P \rightarrow P$ be a function such that for every $X \subseteq Y$, we have $f(X) \subseteq f(Y)$. Show that for some K , $f(K) = K$.
- You are walking from the NE corner of 14th St and 8th Ave to the SW corner of 16th St and 6th Ave. Which direction should you walk first? (Does it even make a difference?)
- Six people go into a closet with 5 blue hats and 5 red hats. Each puts on a hat but doesn't know what color it is. After they come out:
Person #1 says: I don't know what color my hat is.
Person #2 then says: I also don't know what color my hat is.
Person #1 then asks: Do any of you know what color your hat is now?
All 5 other people simultaneously answer: No
If each one has perfect logic and reasoning skills, based on all the information, how many people are now sure what color their hat is?

10. Let $A = (a_{ij})$ be the $n \times n$ matrix with $a_{ij} = 1$ if $i \neq j$, and $a_{ii} = 0$. Show that the number of non-zero terms in the expansion of $\det A$ is $n! \sum_0^n (-1)^i / i!$.
11. Let f be a continuous function on $[0, 1]$. Prove that $\int_0^1 \int_x^1 \int_x^y f(x)f(y)f(z)dzdydx = \frac{1}{6}(\int_0^1 f(x)dx)^3$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.