12. Probability

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1 Classical results

Power of collaboration. There are two students in Mr. Jones' English class. Each of these students usually gets a D (60-69%) on their weekly spelling tests. Mr. Jones announces that next week's spelling test will be taken in pairs. If each of these D students independently has a 69% chance of knowing how to spell each word correctly and knows which words they can and cannot spell (no one will guess how to spell a word they do not know how to spell for sure), what grade will they earn if they collaborate on that spelling test?

2 Problems

1. Your sports team is playing a best-of-15 series against a single opponent. Against this opponent, your team wins with probability 40%, unless it is behind in the series (with strictly fewer wins than losses so far in these 15-game finals), at which point your team wins with probability 60%. What is the probability that your team wins?

2. There's a bag with 1001 marbles, of which 501 are white and 500 are black. You can take any number of them out of the bag. If you take out the same number of black and white marbles, then you earn a number of dollars equal to the number of marbles you took out. What should you do?

3. How do you estimate that an event is a "100-year" occurrence (having 1% chance of happening, in each particular year)? For simplicity, consider a binary event, which either happens or doesn't (as opposed to a real number measurement). Since each year is considered to be independent, this is done by estimating the underlying probability p of a Binomial random variable, and reporting it as a $\frac{1}{p}$-year event. One method uses Maximum Likelihood Estimation, which calculates the value of p which maximizes the probability of the witnessed activity, given the Binomial model.

   Suppose that historical observations for a particular event began in 1901, and in that very first year, the event was witnessed. It was witnessed again in the year 2000. At the end of 2000, if this were reported as an n-year event, what would be the best estimate for n, rounded to the nearest whole number?

4. Suppose that Country A has 400 million people, Country B has 100 million people, and each country always sends its strongest person to the Olympics for weightlifting. The strength of each person is an independent Normal random variable with a common mean and variance, which is the same across both countries. What is the probability that country A wins, rounded to the nearest whole percent?

5. You flip a single coin until you see a tail followed by a head. Zora flips a coin until she sees two heads in a row. Let Y be the expected number of times you end up flipping your coin, and let Z be the expected number of times Zora ends up flipping her coin. What is $Z - Y$?

6. Consider a random walk in which steps of unit length are taken, where each step is in a uniformly random direction, independent of all other steps. What is the expected value of the squared distance from the origin after n steps?

7. Marathon Swimming was introduced as an Olympic event in 2008, with a 10 km race in open water. In total, the race takes about 2 hours and in 2012, the gap between the gold and silver medals was only 0.4 seconds! How could such a long race be so close? Perhaps people swim a bit faster when they are behind, and a bit slower when they are ahead.

   Model this with a random walk along the integers, which starts at 0, and takes steps of $+1$ or -1, where the probability of stepping towards 0 is 0.6, and the probability of stepping away from 0 is 0.4. (When it is at 0, its probability of a $+1$ step is 0.6, and its probability of a -1 step is 0.4.) What is the probability that after 1 billion steps, the walk is farther than 10 steps from the origin? Round your answer to the nearest whole percent.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.