

11. Integer polynomials

Po-Shen Loh

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1 Classical results

Vandermonde determinant. Let a_0, a_1, \dots, a_n be distinct numbers, and let b_0, b_1, \dots, b_n be arbitrary (possibly equal to each other or to any of the a_i). Then there is a unique polynomial $p(x) = c_n x^n + \dots + c_0$ such that $p(a_i) = b_i$ for all $0 \leq i \leq n$.

Lagrange interpolation. An expression for the above polynomial is

$$p(x) = \sum_{i=0}^n \frac{b_i}{\prod_{j \neq i} (a_i - a_j)} \prod_{j \neq i} (x - a_j).$$

Fermat's Last Theorem. The equation $x^n + y^n = z^n$ has no positive integer solutions (x, y, z, n) with $n \geq 3$.

2 Problems

1. Let $p(x)$ be the polynomial $(x - a)(x - b)(x - c)(x - d)$. Assume $p(x) = 0$ has four distinct integral roots and that $p(x) = 4$ has an integral root k . Show that k is the mean of a, b, c, d .
2. Let $p(x)$ be a polynomial with integer coefficients. Suppose that for some positive integer c , none of $p(1), p(2), \dots, p(c)$ are divisible by c . Prove that $p(b)$ is not zero for any integer b .
3. Let $p(x)$ be a real polynomial of degree n such that $p(m)$ is integral for all integers m . Show that if k is a coefficient of $p(x)$, then $n!k$ is an integer.
4. Find all rational triples (a, b, c) for which a, b, c are the roots of $x^3 + ax^2 + bx + c = 0$.
5. For what positive integers n does the polynomial $p(x) = x^n + (2+x)^n + (2-x)^n$ have a rational root?
6. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, and suppose that each a_n is 0 or 1.
 - (1) Show that if $f(1/2)$ is rational, then $f(x)$ has the form $p(x)/q(x)$ for some integer polynomials $p(x)$ and $q(x)$.
 - (2) Show that if $f(1/2)$ is not rational, then $f(x)$ does not have the form $p(x)/q(x)$ for any integer polynomials $p(x)$ and $q(x)$.
7. Let n be a nonzero integer. Prove that $n^4 - 7n^2 + 1$ can never be a perfect square.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.