4. Calculus

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1 Classical results

1. Let $f : [0,1] \to \mathbb{R}$ be a monotone increasing function, and let $g : [0,1] \to \mathbb{R}$ be a monotone decreasing function. Show that $\int_0^1 f(x)g(x)dx \le \int_0^1 f(x)dx \int_0^1 g(x)dx$, i.e., that the expected value of the product of two negatively correlated random variables is at most the product of their expected values.

2 Problems

1. Given functions $f, g : \mathbb{R} \to \mathbb{R}$, and $x \in \mathbb{R}$, let $I(fg)$ denote the function which maps x to $\int_1^x f(t)g(t)dt$. Prove that whenever $a(x)$, $b(x)$, $c(x)$, and $d(x)$ are real polynomials, the polynomial

$$I(ac)I(bd) - I(ad)I(bc)$$

is divisible by $(x-1)^4$.

2. Let S be a spherical shell of radius 1, i.e., the set of points satisfying $x^2 + y^2 + z^2 = 1$. Find the average straight line distance between two points of S.

3. Let $p(x)$ be a real polynomial of degree at most 2, which satisfies $|p(x)| \le 1$ for all $-1 \le x \le 1$. Show that $|p'(x)| \le 4$ for all $-1 \le x \le 1$.

4. Let K be a positive real number, and let $f : [0,1] \to \mathbb{R}$ be a differentiable function whose derivative satisfies $|f'(x)| \le K$ for all $0 \le x \le 1$. Prove that

$$\left| \int_0^1 f(x)dx - \sum_{i=1}^{n} \frac{f(i/n)}{n} \right| \le \frac{K}{n}\,.$$

5. Let $f : [0,1] \to \mathbb{R}^+$ be a monotone decreasing continuous function. Show that

$$\int_0^1 f(x)dx \int_0^1 xf(x)^2dx \le \int_0^1 xf(x)dx \int_0^1 f(x)^2dx\,.$$

6. Let $f : [0,1] \to \mathbb{R}$ be a continuous function which satisfies $\int_0^1 x^n f(x)dx = 0$ for all non-negative integers n. Prove that f is the zero function.

7. Let $f : [1,\infty) \to \mathbb{R}$ be a differentiable function which satisfies $f'(x) = \frac{1}{x^2 + f(x)^2}$ and $f(1) = 1$. Show that as $x \to \infty$, $f(x)$ tends to a limit which is less than $1 + \frac{\pi}{4}$.

8. Show that there is at most one continuous function $f : [0,1]^2 \to \mathbb{R}$ satisfying $f(x,y) = 1 + \int_0^x \int_0^y f(s,t)dtds$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.