3. Number theory

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1 Classical results

1. $\sqrt{6}$ is irrational.

2. For any irrational number α, the fractional parts of its integer multiples are dense in $[0, 1)$. That means that for any $\epsilon > 0$ and any real number $0 \le r < 1$, there is an integer z so that the fractional part $\{z\alpha\} = z\alpha - \lfloor z\alpha \rfloor$ is within ϵ of r. The same is not true when α is rational.

3. It is still an open question to determine whether $e + \pi$ is rational. It is also still open to determine whether $e \cdot \pi$ is rational. However, it is known that at least one of them is irrational.

4. Find all integer solutions to the equation $\frac{2}{x} + \frac{8}{y} = 1$.

2 Problems

1. Prove that if a, b, c are integers and $a\sqrt{2} + b\sqrt{3} + c = 0$, then $a = b = c = 0$.

2. Find all integral x and y satisfying the equation $2\sqrt{6} + 5\sqrt{10} = \sqrt{x} + \sqrt{y}$.

3. Welcome to the 2016-2017 school year! A brand new school has installed exactly 2017 lockers, numbered from 1 to 2017, running side by side all the way around its perimeter so that locker #2017 is right next to locker #1. After checking the lockers, all of the odd numbered ones were left open, and all of the even numbered ones were shut.

   A prankster starts at locker #1, and flips its state from open to shut. He then moves one locker to the left (to #2017), and flips its state from open to shut. He then moves three more lockers to the left (to #2014), and flips its state from shut to open. He then moves five more lockers to the left (to #2009), and flips its state from open to shut. He keeps going in this way, until he has flipped a total of 2017 lockers.

   How many lockers are open after he is finished?

4. Given any positive integer n, show that we can find a positive integer m such that mn uses all ten digits when written in the usual base 10.

5. Show that for any positive integer r, we can find integers m, n such that $m^2 - n^2 = r^3$.

6. Let n be a positive integer. Prove that $n(n + 1)(n + 2)(n + 3)$ cannot be a square or a cube.

7. Prove that there are only finitely many cuboidal blocks with integer sides $a \times b \times c$, such that if the block is painted on the outside and then cut into unit cubes, exactly half the cubes have no face painted.

8. α and β are positive irrational numbers satisfying $1/\alpha + 1/\beta = 1$. Let $a_n = \lfloor n\alpha \rfloor$ and $b_n = \lfloor n\beta \rfloor$, for $n = 1, 2, 3, \ldots$. Show that the sequences a_n and b_n are disjoint and that every positive integer belongs to one or the other.

9. If x is a positive rational, show that we can find distinct positive integers $a_1, a_2, \ldots, a_n$ such that $x = \sum 1/a_i$.

10. Show that we can express any irrational number $0 < \alpha < 1$ uniquely in the form $\sum_1^\infty (-1)^{n+1} 1/(a_1 a_2 \cdots a_n)$, where a_i is a strictly monotonic increasing sequence of positive integers. Find a_1, a_2, a_3 for $\alpha = 1/\sqrt{2}$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.