2. Polynomials

Po-Shen Loh

CMU Putnam Seminar, Fall 2020

1 Classical results

1. Find a nice expression for the derivative of the polynomial $(x-1)(x-2)(x-3)^2$.
2. Let $p(x) = a_n x^n + \cdots + a_0$ be a polynomial which satisfies $p(-x) = p(x)$ for every real x. Prove that $a_i = 0$ for every odd i.

2 Problems

1. Show that the real polynomial $\sum_0^n a_i x^i$ has at least one real root if $\sum \frac{a_i}{i+1} = 0$.
2. Prove that we can find a real polynomial $p(y)$ such that $p(x - 1/x) = x^n - 1/x^n$ (where n is a positive integer) iff n is odd.
3. The roots of $x^3 + ax^2 + bx + c = 0$ are α, β, and γ. Find the cubic whose roots are α^3, β^3, and γ^3.
4. Let $p(x)$ be a polynomial with real coefficients, and let $r(x)$ be the polynomial defined by the derivative $r(x) = p'(x)$. Suppose that there are positive integers a and b for which $r^a(x)$ divides $p^b(x)$ as polynomials. Prove that for some real numbers A and α, and for some integer n, we have $p(x) = A(x-\alpha)^n$.
5. $p(z) = z^2 + az + b$ has complex coefficients. $|p(z)| = 1$ on the unit circle $|z| = 1$. Show that $a = b = 0$.
6. Let $p(z)$ and $q(z)$ be complex nonconstant polynomials with the same set of roots (but possibly different multiplicities). Suppose that $p(z)+1$ and $q(z)+1$ also have the same set of roots. Show that $p(z) = q(z)$.
7. Let $p(z)$ be a polynomial of degree n with complex coefficients. Its roots (in the complex plane) can be covered by a disk of radius r. Show that for any complex k, the roots of $np(z) - kp'(z)$ can be covered by a disk of radius $r + |k|$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.