11. Integer Polynomials
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1 Classical results

Well-known fact. Let $P(n)$ be a polynomial with integer coefficients, and let $a$ and $b$ be integers. Show that $P(a) - P(b)$ is divisible by $a - b$.

Gauss. If a polynomial with integer coefficients can be factored into a product of polynomials with rational coefficients, then it can also be factored into a product of polynomials with integer coefficients.

Eisenstein. Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial, such that there is a prime $p$ for which

(i) $p$ divides each of $a_0, a_1, \ldots, a_{n-1},$
(ii) $p$ does not divide $a_n$, and
(iii) $p^2$ does not divide $a_0$.

Then $P(x)$ cannot be expressed as the product of two non-constant polynomials with integer coefficients.

Integers. There is a polynomial which takes integer values at all integer points, but does not have integer coefficients.

Rational Root Theorem. Suppose that $P(x) = a_nx^n + \cdots + a_0$ is a polynomial with integer coefficients, and that one of the roots is the rational number $p/q$ (in lowest terms). Then, $p \mid a_0$ and $q \mid a_n$.

2 Problems

1. What is the largest positive integer that is a factor of $P(1) - 2P(7) + P(13)$, for every polynomial $P$ with integer coefficients?

2. Find a nonzero polynomial $P(x, y)$ such that $P([a], [2a]) = 0$ for all real numbers $a$. (Note: $[\nu]$ is the greatest integer less than or equal to $\nu$.)

3. Prove that for every prime number $p$, the polynomial

$$P(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

cannot be expressed as the product of two non-constant polynomials with integer coefficients.

4. Suppose that the polynomial $P(x)$ with integer coefficients takes values $\pm 1$ at three different integer points. Prove that it has no integer zeros.

5. Let $P(x)$ be a polynomial with integer coefficients. Suppose that there is an integer $a$ for which $P(P(\cdots P(a) \cdots)) = a$, where $P$ is iterated some number of times which is at least twice. Then, $P(P(a)) = a$. 

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6. Let \( P(x) \) be a polynomial with integer coefficients which cannot be factored as a product of polynomials with integer coefficients. Prove that \( P(x) \) has no multiple roots.

7. Let \( P(x) = x^n + 5x^{n-1} + 3 \), where \( n > 1 \) is an integer. Prove that \( P(x) \) cannot be expressed as the product of two non-constant polynomials with integer coefficients.

8. Suppose \( q_0, q_1, q_2, \ldots \) is an infinite sequence of integers satisfying the following two conditions:
   
   (i) \( m - n \) divides \( q_m - q_n \) for \( m > n \geq 0 \),
   
   (ii) there is a polynomial \( P \) and an integer \( \Delta \) such that \( |q_n - P(n)| < \Delta \) for all \( n \).

   Prove that there is a polynomial \( Q \) such that \( q_n = Q(n) \) for all \( n \).

9. For every polynomial \( P(x) \) with integer coefficients, does there always exist a positive integer \( k \) such that \( P(x) - k \) is irreducible over integers?

10. Let \( n \) be a positive integer, and let \( p(x) \) be a polynomial of degree \( n \) with integer coefficients. Prove that

\[
\max_{0 \leq x \leq 1} |p(x)| > \frac{1}{e^n}
\]