Putnam E.12

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1 Problems

Putnam 2011/A1. Define a *growing spiral* in the plane to be a sequence of points with integer coordinates $P_0 = (0, 0), P_1, \ldots, P_n$ such that $n \geq 2$ and:

- the directed line segments $P_0P_1, P_1P_2, \ldots, P_{n-1}P_n$ are in the successive coordinate directions east (for P_0P_1), north, west, south, east, etc.;
- the lengths of these line segments are positive and strictly increasing.

How many of the points (x, y) with integer coordinates $0 \leq x \leq 2011, 0 \leq y \leq 2011$ *cannot* be the last point, P_n of any growing spiral?

Putnam 2011/A2. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \ldots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \ldots a_n}$$

converges, and evaluate S.

Putnam 2011/A3. Find a real number c and a positive number L for which

$$\lim_{r \to \infty} \frac{r^c \int_0^{\pi/2} x^r \sin x \, dx}{\int_0^{\pi/2} x^r \cos x \, dx} = L.$$