13.

(Just do it)

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1 Famous results

There are countably many rational numbers. $|\mathbb{Q}| = |\mathbb{N}|$.

Cantor-Bernstein-Schroeder. If A and B are (possibly infinite) sets such that there exist injective functions $f : A \to B$ and $g : B \to A$, then there is a bijection $h : A \to B$.

2 Problems

1. Let S be a subset of the rational numbers which is closed under addition and multiplication. Suppose that $0 \notin S$, and for every nonzero rational number x, exactly one of $\{x, -x\}$ is in S. Prove that S is the set of all positive rationals.

2. Find the set of positive integers with sum 1979 and maximum possible product.

3. A ladder has n rungs. On each rung, a blob of red paint is stuck to either one or the other endpoint of that rung. The ladder is then twisted into a Möbius strip. A mouse starts walking along the edge, and counts the number of inter-rung segments which have both endpoints red. Prove that the resulting number always has opposite parity to n.

4. Let $2^{\mathbb{N}}$ denote the set of all subsets of $\mathbb{N}$. Find a function $f : \mathbb{R} \to 2^{\mathbb{N}}$ which satisfies the property that for all $x, y \in \mathbb{R}$, $x < y \Rightarrow f(x) \subset f(y)$ (and $f(x) \neq f(y)$).

5. Let X be a finite set, and let $f : X \to X$ be a bijection. Prove that there are functions $g : X \to X$ and $h : X \to X$ such that $f = gh$, $gg = 1$, and $hh = 1$, where 1 denotes the identity function $1(x) = x$.

6. If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$. A game is finite if with probability 1 it must end in a finite number of moves.)

7. (Looks simple, but harder.) Suppose that a standard, fair, 6-sided die is rolled 10 times. The sum of the numbers rolled will be somewhere between 10 and 60 inclusive. What is the probability that the sum will be divisible by 5?

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.