11. Integer polynomials

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1 Famous results

Divisibility. If a and b are integers, and $p(x)$ is a polynomial with integer coefficients, then $p(a) - p(b)$ is always divisible by $a - b$.

Chinese remainder theorem. Let $m_1, m_2, \ldots, m_k$ be positive integers which are pairwise relatively prime, and let $a_1, \ldots, a_k$ be arbitrary integers. Then, the following system has integer solutions for x:

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ &\vdots \\ x &\equiv a_k \pmod{m_k}, \end{aligned}$$

and all solutions x have the same residue modulo the product $m_1 m_2 \cdots m_k$.

Gauss's lemma. Non-constant integer polynomials which are irreducible over $\mathbb{Z}$ are also irreducible over $\mathbb{Q}$.

Eisenstein's criterion. Suppose that $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, and there is a prime number p such that (i) p divides each of a_0, a_1, ..., a_{n-1}, (ii) p does not divide a_n, and (iii) p^2 does not divide a_0. Then, $f(x)$ is irreducible over $\mathbb{Q}$.

2 Problems

1. Albert Einstein and Homer Simpson are playing a game in which they are creating a polynomial

$$p(x) = x^{2012} + a_{2011}x^{2011} + \cdots + a_1 x + a_0 .$$

They take turns choosing one of the coefficients $a_0, \ldots, a_{2011}$, assigning a real value to it (even though the topic of this week is integer polynomials). Once a value is assigned to a coefficient, it cannot be overwritten in a future turn, and the game ends when all coefficients have been assigned. Albert moves first. Homer's goal is to make $p(x)$ divisible by a fixed polynomial $m(x)$, and Albert's goal is to prevent this.

   (a) Which of the players has a winning strategy if $m(x) = x - 2012$?

   (b) What if $m(x) = x^2 + 1$?

2. Let p, q, and s be nonconstant integer polynomials such that $p(x) = q(x)s(x)$. Suppose that the polynomial $p(x) - 2008$ has at least 81 distinct integer roots. Prove that the degree of q must be greater than 5.

3. Let p be a quadratic polynomial with integer coefficients. Suppose that $p(z)$ is divisible by 5 for every integer z. Prove that all coefficients of p are divisible by 5.

4. Let x, y, z be integers such that $x^4 + y^4 + z^4$ is divisible by 29. Prove that $x^4 + y^4 + z^4$ is actually divisible by 29^4.

5. Let p be a polynomial with integer coefficients, and let $a_1, \ldots, a_k$ be distinct integers. Prove that there always exists an $a \in \mathbb{Z}$ such that $p(a)$ is an integer multiple of $p(a_i)$ for all i.

6. Let $f(x)$ be a rational function, i.e., there are polynomials p and q such that $f(x) = p(x)/q(x)$ for all x. Prove that if $f(n)$ is an integer for infinitely many integers n, then f is actually a polynomial.

7. Let a, b be integers. Show that the set $\{ax^2 + by^2 : x, y \in \mathbb{Z}\}$ misses infinitely many integers.

8. Let a, b be integers. Show that the set $\{ax^5 + by^5 : x, y \in \mathbb{Z}\}$ misses infinitely many integers.

9. Let a, b, n be integers (n positive) for which the set $\{ax^n + by^n : x, y \in \mathbb{Z}\}$ includes all but finitely many integers. Prove that $n = 1$.

10. Let p be a polynomial with real coefficients and degree n. Suppose that $\frac{p(b)-p(a)}{b-a}$ is an integer for all $0 \leq a < b \leq n$. Prove that $\frac{p(b)-p(a)}{b-a}$ is an integer for all pairs of distinct integers $a < b$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.