13. Basic methods

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1 Classical results

Helly. Let $C_1, C_2, \ldots, C_n$ be a collection of convex subsets of $\mathbb{R}^d$, with the property that every $d+1$ of them have nonempty intersection. Then the whole collection has nonempty intersection.

Brouwer's Fixed Point Theorem. Every continuous function from a closed Euclidean ball to itself has a fixed point.

2 Problems

1. (1913 entrance exam to Carnegie Institute of Technology: Math.) A spherical triangle has angles of $70°$, $90°$, and $100°$, and the underlying sphere has radius 10. What is the area of the spherical triangle?

2. (1913 entrance exam to CIT: English.) What is the feminine form of the noun "duck"?

3. Six people all go into a dark closet, which they know contains 10 hats which are perfectly identical except that 5 are blue and 5 are red. They each put on a hat but they dont know what color their own hat is. After they all come out of the closet, they try to deduce what color their own hat is, by looking at one another.

   One person says: "I dont know what color my hat is."

   Another person says: "I also dont know what color my hat is."

   The first person asks out loud: "Do any of you know what color your hat is now?"

   All five of the others simultaneously answer "No."

   If each one has perfect logic and reasoning skills, based on all the information, how many people are now sure what color their hat is?

4. Show that there are exactly $\binom{n-k+1}{k}$ subsets of $\{1, 2, \ldots, n\}$ with k elements and not containing both i and $i+1$ for any i.

5. Does the series $\sum_{n=2}^{\infty} \frac{1}{\log(n!)}$ converge?

6. A finite set of circles divides the plane into regions. Show that we can color the plane with two colors so that no two adjacent regions (with a common arc of non-zero length forming part of each region's boundary) have the same color.

7. Let S be a finite collection of closed intervals on the real line such that any two have a point in common. Prove that the intersection of all the intervals is non-empty.

8. Let S be a set and P the set of all subsets of S. Let $f : P \to P$ be a function such that for every $X \subseteq Y$, we have $f(X) \subseteq f(Y)$. Show that for some K, $f(K) = K$.

9. Let $A = (a_{ij})$ be the $n \times n$ matrix with $a_{ij} = 1$ if $i \neq j$, and $a_{ii} = 0$. Show that the number of non-zero terms in the expansion of $\det A$ is $n! \sum_0^n (-1)^i / i!$.

10. Let f be a continuous function on $[0, 1]$. Prove that $\int_0^1 \int_x^1 \int_x^y f(x)f(y)f(z)dzdydx = \frac{1}{6}(\int_0^1 f(x)dx)^3$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.