

10. Combinatorics

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CMU Putnam Seminar, Fall 2016

1 Classical results

Erdős-Ko-Rado. Let \mathcal{F} be a family of k -element subsets of $\{1, 2, \dots, n\}$, with the property that every pair of members of \mathcal{F} has nonempty intersection. Then the size of \mathcal{F} is at most $\binom{n-1}{k-1}$.

Lucas. Let n and k be non-negative integers, with base- p expansions $n = (n_t n_{t-1} \dots n_0)_{(p)}$ and $k = (k_t k_{t-1} \dots k_0)_{(p)}$, respectively. Then

$$\binom{n}{k} \equiv \binom{n_t}{k_t} \times \binom{n_{t-1}}{k_{t-1}} \times \dots \times \binom{n_0}{k_0} \pmod{p}.$$

2 Problems

1. Let X be a subset of $\{1, 2, 3, \dots, 2n\}$ with $n + 1$ elements. Show that we can find $a, b \in X$ with a dividing b .
2. Given any five points in the interior of a square side 1, show that two of the points are a distance apart less than $k = \frac{1}{\sqrt{2}}$. Is this result true for a smaller k ?
3. Let S be a finite set, and suppose that a collection \mathcal{F} of subsets of S has the property that any two members of \mathcal{F} have at least one element in common, but \mathcal{F} cannot be extended (while keeping this property). Prove that \mathcal{F} contains just half of the subsets of S .
4. Show that the number of ways of representing n as an ordered sum of 1's and 2's equals the number of ways of representing $n+2$ as an ordered sum of integers greater than 1. For example: $4 = 1+1+1+1 = 2+2 = 2+1+1 = 1+2+1 = 1+1+2$ (5 ways) and $6 = 4+2 = 2+4 = 3+3 = 2+2+2$ (5 ways).
5. Show that for any given positive integer n , the number of odd $\binom{n}{m}$ with $0 \leq m \leq n$ is a power of 2.
6. A graph has n vertices $\{1, 2, \dots, n\}$ and a complete set of edges. Each edge is oriented, as either $i \rightarrow j$ or $j \rightarrow i$. Show that we can find a permutation of the vertices a_i so that $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \dots \rightarrow a_n$.
7. Let a_1, a_2, \dots, a_n be a permutation of the integers $1, \dots, n$. Call a_i a “big” integer if $a_i > a_j$ for all $j > i$. Find the mean number of “big” integers over all permutations on the first n integers.
8. In a tournament of n players, every pair of players plays once. There are no draws. Player i wins w_i games. Prove that we can find three players i, j, k such that i beats j , j beats k and k beats i iff $\sum_{i=1}^n w_i^2 < \frac{(n-1)n(2n-1)}{6}$.
9. Let n be a positive integer. Suppose we have an infinite sequence of 0's and 1's is such that it only contains n different blocks of n consecutive terms. Show that it is eventually periodic.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.