# 13. Analysis

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## 1 Strange, but true

**Very not continuous.** There is a nowhere continuous function whose absolute value is everywhere continuous.

**Very not bounded.** There is a function that is everywhere finite but everywhere locally unbounded.

**Periodicity.** There is a nonconstant periodic function which has no smallest positive period.

**Inverse continuity.** There is a function which is continuous and injective from an interval to a range which is not necessarily a subset of $\mathbb{R}$, but whose inverse is not continuous.

**Continuity and rationals.** There is a function which is continuous at every irrational point and discontinuous at every rational point.

**Monotonicity.** There is a continuous function which is nowhere monotonic.

**Hydra.** There is a function with domain $[0, 1]$ whose range for every nondegenerate subinterval of $[0, 1]$ is $[0, 1]$.

**Linearity.** There is a discontinous linear function (function which satisfies $f(x+y) = f(x)+f(y)$ for every $x, y \in \mathbb{R}$).

**Joint continuity.** There is a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is not continuous, but has the properties that (i) for every $x_0 \in \mathbb{R}$, the function $f_{x_0} : \mathbb{R} \to \mathbb{R}$ defined by $f_{x_0}(t) = f(x_0, t)$ is continuous, and (ii) for every $y_0 \in \mathbb{R}$, the function $f_{y_0} : \mathbb{R} \to \mathbb{R}$ defined by $f_{y_0}(t) = f(t, y_0)$ is continuous.

## 2 More normal problems

1. Let $X$ be the unit square $[0, 1] \times [0, 1]$, and let $f$ be a continuous function from $X$ to $\mathbb{R}$. Suppose that $\int_Y f(x, y)dxdy = 0$ for all squares $Y$ for which (i) $Y \subset X$, (ii) the sides of $Y$ are parallel to those of $X$, and (iii) at least one of $Y$'s sides is contained in the boundary of $X$. Is it true that $f(x, y) = 0$ for all $(x, y) \in X$?

2. Given that $f(x)$ increases from 0 to 1 as $x$ does, prove that the graph of $y = f(x)$ between $0 \le x \le 1$ can be covered by $n$ rectangles with sides parallel to the axes and each having area $1/n^2$.

3. Show that $\lim_{x\to\infty} \frac{f(x)}{x^2} = 1$ does not imply $\lim_{x\to\infty} \frac{f'(x)}{2x} = 1$, but it does for convex $f$.

4. Let $f(x)$ be a continuous function which satisfies

$$\lim_{h\to 0^+} \frac{f(x + 2h) - f(x + h)}{h} = 0$$

for each $x$. Prove that $f(x)$ is a constant.

5. Show that $f(x) \in C^1[a,b]$ iff the limit as $h \to 0$ of

$$\frac{f(x+h) - f(x)}{h}$$

exists uniformly on $[a,b]$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function with the properties that (i) for every $x_0 \in \mathbb{R}$, the function $f_{x_0} : \mathbb{R} \to \mathbb{R}$ defined by $f_{x_0}(t) = f(x_0, t)$ is continuous, and (ii) for every $y_0 \in \mathbb{R}$, the function $f_{y_0} : \mathbb{R} \to \mathbb{R}$ defined by $f_{y_0}(t) = f(t, y_0)$ is continuous. Show that there is a sequence of continuous functions $g_n : \mathbb{R}^2 \to \mathbb{R}$ which tends to $f$ pointwise.

# 3 Homework

Please write up solutions to two of the statements/problems, to turn in at next week's meeting. One of them may be a problem that we solved in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.