Putnam E.1

Po-Shen Loh

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1 Problems

Putnam 1987/A1. Curves A, B, C, and D are defined in the plane as follows:[1]

$$
\begin{aligned}
A &= \left\{(x, y) : x^2 - y^2 = \frac{x}{x^2 + y^2}\right\}, \\
B &= \left\{(x, y) : 2xy + \frac{y}{x^2 + y^2} = 3\right\}, \\
C &= \left\{(x, y) : x^3 - 3xy^2 + 3y = 1\right\}, \\
D &= \left\{(x, y) : 3x^2y - 3x - y^3 = 0\right\}.
\end{aligned}
$$

Prove that $A \cap B = C \cap D$.

Putnam 1987/A2. The sequence of digits

$$123456789101112131415161718192021\ldots$$

is obtained by writing the positive integers in order. If the 10^n-th digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

Putnam 1987/A3. For all real x, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x .$$

(a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.

(b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.

[1] The equations defining A and B are indeterminate at $(0, 0)$. The point $(0, 0)$ belongs to neither.