14. Calculus and Linear Algebra

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CMU Putnam Seminar, Fall 2012

1 Warm-up

Putnam 2012/A0. When and where is the Putnam?

1913 entrance exam to Carnegie Institute of Technology (Math). A spherical triangle has angles of $70°$, $90°$, and $100°$, and the underlying sphere has radius 10. What is the area of the spherical triangle?

1913 entrance exam to CIT (English). What is the feminine form of the noun "duck"?

2 Problems

Putnam 1941/A2. Define $f(x) = \int_0^x \sum_{i=0}^{n-1} \frac{(x-t)^i}{i!} dt$. Calculate the n-th derivative $f^{(n)}(x)$.

Putnam 1942/A3. Does $\sum_{n \geq 0} \frac{n! k^n}{(n+1)^n}$ converge or diverge for $k = \frac{19}{7}$?

Putnam 1941/B3. Let y_1 and y_2 be any two linearly independent solutions to the differential equation $y'' + p(x)y' + q(x)y = 0$. Let $z = y_1 y_2$. Find the differential equation satisfied by z.

Putnam 1955/B2. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function, with f'' continuous and $f(0) = 0$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x)/x$ for $x \neq 0$, and $g(0) = f'(0)$. Show that g is differentiable and that g' is continuous.

Putnam 1949/A6. Show that $\prod_{n=1}^{\infty} \frac{1+2\cos(2z/3^n)}{3} = \frac{\sin z}{z}$ for all complex z.

Putnam 1948/B6. Take the origin O of the complex plane to be the vertex of a cube, so that OA, OB, OC are edges of the cube (with A, B, C possibly lying in the third dimension, outside the complex plane). Let the feet of the perpendiculars from A, B, C to the complex plane be the complex numbers u, v, w. Show that $u^2 + v^2 + w^2 = 0$.

Putnam 1948/A5. Let $\omega_1, \omega_2, \ldots, \omega_n$ be the n-th roots of unity. Find

$$\prod_{i<j} (\omega_i - \omega_j)^2 .$$

Putnam 1940/B6. The $n \times n$ matrix (m_{ij}) is defined as $m_{ij} = a_i a_j$ for $i \neq j$, and $a_i^2 + k$ for $i = j$. Show that $\det(m_{ij})$ is divisible by k^{n-1} and find its other factor.

3 No homework

Please do not submit write-ups for any problems. There is no homework for next week. There is no next week. Do not pass Go, do not collect $200.