13. Basic methods

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1 Classical results

Helly. Let $C_1, C_2, \ldots, C_n$ be a collection of convex subsets of $\mathbb{R}^d$, with the property that every $d+1$ of them have nonempty intersection. Then the whole collection has nonempty intersection.

Brouwer's Fixed Point Theorem. Every continuous function from a closed Euclidean ball to itself has a fixed point.

2 Problems

1913 entrance exam to Carnegie Institute of Technology (Math). A spherical triangle has angles of 70°, 90°, and 100°, and the underlying sphere has radius 10. What is the area of the spherical triangle?

1913 entrance exam to CIT (English). What is the feminine form of the noun "duck"?

Putnam 1956/A5. Show that there are exactly $\binom{n-k+1}{k}$ subsets of $\{1, 2, \ldots, n\}$ with k elements and not containing both i and $i+1$ for any i.

Putnam 1950/A2. Does the series $\sum_{n=2}^{\infty} \frac{1}{\log(n!)}$ converge?

Putnam 1964/B4. A finite set of circles divides the plane into regions. Show that we can color the plane with two colors so that no two adjacent regions (with a common arc of non-zero length forming part of each region's boundary) have the same color.

Putnam 1954/B3. Let S be a finite collection of closed intervals on the real line such that any two have a point in common. Prove that the intersection of all the intervals is non-empty.

Putnam 1957/B5. Let S be a set and P the set of all subsets of S. Let $f : P \to P$ be a function such that for every $X \subseteq Y$, we have $f(X) \subseteq f(Y)$. Show that for some K, $f(K) = K$.

Putnam 1958/A5. Let $A = (a_{ij})$ be the $n \times n$ matrix with $a_{ij} = 1$ if $i \neq j$, and $a_{ii} = 0$. Show that the number of non-zero terms in the expansion of $\det A$ is $n! \sum_0^n (-1)^i / i!$.

Putnam 1941/B6. Let f be a continuous function on $[0, 1]$. Prove that $\int_0^1 \int_x^1 \int_x^y f(x)f(y)f(z)dzdydx = \frac{1}{6}(\int_0^1 f(x)dx)^3$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.