1. Proof by contradiction

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1 Classical results

1. The real number $\sqrt{2}$ is irrational.

2. The set of real numbers is uncountable.

2 Problems

Putnam 1952/A1. The polynomial $p(x)$ has all integral coefficients. The leading coefficient, the constant term, and $p(1)$ are all odd. Show that $p(x)$ has no rational roots.

Putnam 1962/A6. X is a subset of the rationals which is closed under addition and multiplication, and it does not contain 0. For any rational $x \neq 0$, exactly one of x or $-x$ is in X. Show that X is the set of all positive rationals.

Putnam 1965/B5. Show that a graph with $2n$ points and $n^2 + 1$ edges necessarily contains a 3-cycle, but that we can find a graph with $2n$ points and n^2 edges without a 3-cycle. A 3-cycle is a collection of 3 vertices x, y, z, such that xy, yz, and zx are all edges in the graph.

Putnam 1952/B6. A, B, C are points of a fixed ellipse E. Show that the area of ABC is maximized if and only if the centroid of ABC is at the center of E. The centroid of a triangle is its center of mass, which also happens to lie at the intersection of its three medians.

Putnam 1964/B6. D is a disk. Show that we cannot find congruent sets A, B with $A \cap B = \emptyset$, and $A \cup B = D$. More formally, D is the closed unit disk, including boundary, i.e., all points (x, y) satisfying $x^2 + y^2 \leq 1$. We must show that it is impossible to choose a subset A of D such that there is a geometric transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is a bijection from A to $D \setminus A$. A geometric transformation is a composition of rotations, translations, and reflections.

Putnam 1958/B5. S is an infinite set of points in the plane. The distance between any two points of S is integral. Prove that S is a subset of a straight line.

Putnam 1964/A6. S is a finite set of collinear points. Let k be the maximum distance between any two points of S. Given a pair of points of S a distance $d < k$ apart, we can find another pair of points of S also a distance d apart. Prove that if two pairs of points of S are distances a and b apart, then a/b is rational.

Putnam 1964/B3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, such that for each $\alpha > 0$, $\lim_{n\to\infty} f(n\alpha) = 0$. (That limit corresponds to sending evaluating $f(\alpha), f(2\alpha), f(3\alpha), \ldots$ and finding the limit of the sequence.) Prove that $\lim_{x\to\infty} f(x) = 0$, where now this limit corresponds to sending x to ∞ along the real axis. That is, for every $\epsilon > 0$, there is a T such that for all real numbers $x > T$, we have $|f(x)| < \epsilon$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.