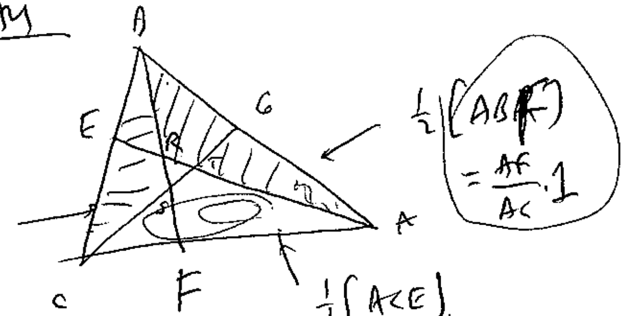
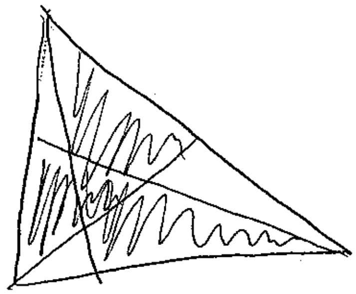
~~2010-10~~
2011-10-08 (A)

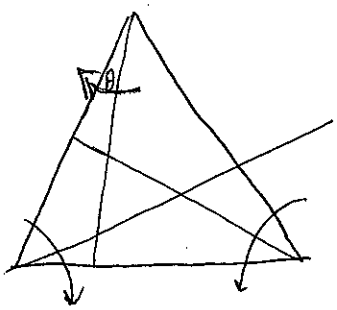
2001/A4

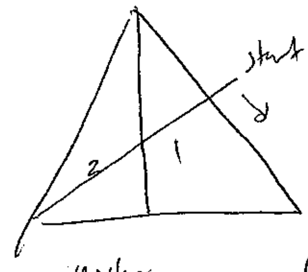


So take whole (1)
Minus those summed

So $1 - \frac{1}{2}\left[\frac{AF}{AC} + \frac{CE}{CB} + \frac{BG}{BA}\right]$.



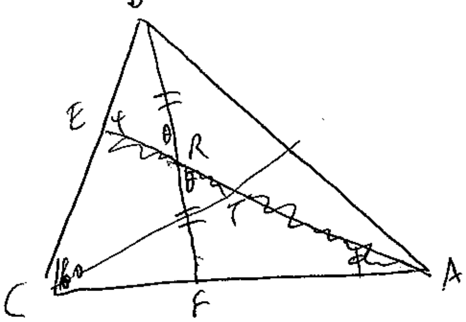




so unique

$\frac{1}{2} \quad \frac{\sqrt{3}}{2}\frac{\cos\phi}{\sin\phi}$

$= \cos 60 +$



$$\frac{AF}{\sin\theta} = \frac{RF}{\sin\phi}.$$

$$\frac{BE}{\sin\theta} = \frac{BR}{\sin\psi}.$$

div

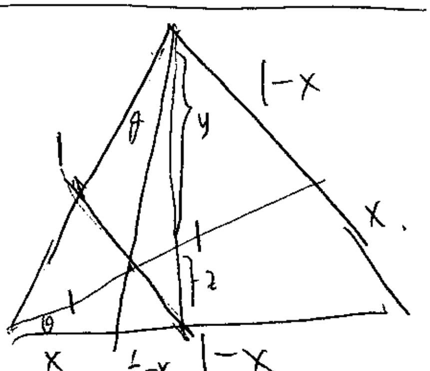
$$\frac{AF}{BE} = \frac{\sin\psi}{\sin\phi} = \frac{\sin(\phi+60)}{\sin\phi}.$$

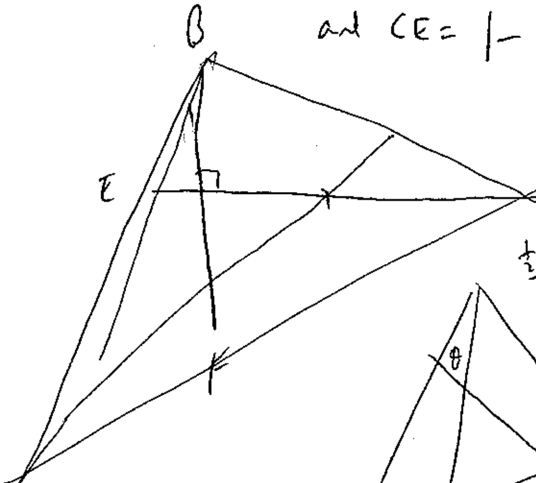
So $BE = AF \frac{\sin(\phi+60)}{\sin\phi}$,

and $CE = 1 - AF\frac{\sin(\phi+60)}{\sin\phi}$.

$[ACE] = 1 - [ABE]$

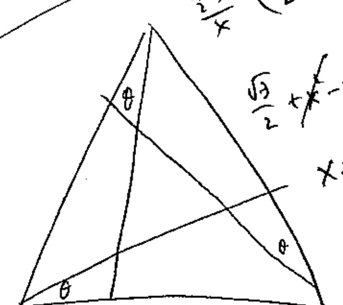
$= 1 - \frac{RE}{AE}\frac{1}{2}[ABF]$





$\frac{\sqrt{3}-x}{x} = 1$

$\frac{\frac{1}{2}-x}{x}$



$\frac{\sqrt{3}}{2} + \cancel{x} - x\left(\frac{1}{2}+\sqrt{3}\right) = \cancel{x}$

$x = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}+\sqrt{3}}$

$= \frac{\sqrt{3}}{1+2\sqrt{3}}$

$= \frac{\sqrt{3}(2\sqrt{3}-1)}{11}$

$= \frac{6-\sqrt{3}}{11}$

$\frac{\frac{1}{2}-x}{x} \cdot \frac{y}{z} = 1$

$\frac{z}{x} = \frac{1}{2}$. $z = \frac{1}{2}x$

$y = \frac{\sqrt{3}}{2} - \frac{1}{2}x$

$1-x = \frac{\sqrt{3}+5}{11}$

2001/A5

2011-10-09

(B)

$n=1.$ $a^2-(a+1)=2001$

$a^2-a-1=2001$

$a^2-a-2002=0.$

$2002=2\times7\times11\times13$

Want diff of 1. No good.

$a^3-(a+1)^2=2001.$

Approx. a^{n+1}.

$a^3-a^2-2a-2002=0.$

$11^3=1331.$ $16^3=2^{12}=4096.$

$a=13$ is solution. Check ok.

$12^3=$ 144 × 12 = 288 + 1440 = 1728

22

169 × 13 = 507 + 1690 = 2197.

$15^3=$ 225 × 15

too big

$2197-196=2001.$

13^3-14^2

$a^4-(a+1)^3=2001?$ $\Rightarrow a\mid 2002.$ $a\mid 2\times7\times11\times13.$

$\vdots$

$2^{n+1}-3^{n}=2001.$

~~$(a+1)^n$~~

$a^{12}-(a+1)^{11}=2001?$

$2\cdot2^n-3^n=2001.$

$a=1$ is impossible

$a=2:$ $2^{12}-3^{11}$

$a=3$ $3^{12}-4^{11}$

Say a even. Then $(\text{even})^{n+1}-3^n\ldots$

Say $11\mid a$. Then ok.

$77^{n+1}-78^n=2001.$

a^{n+1} vs $(a+1)^n$

as $n\nearrow$, (1) $x=a$ (2) $x=a+1$. so (1)-(2) $\searrow$. $77^{77}-78^{76}.$

$a^{n+1}-a^n-na^{n-1}-\binom{n}{2}a^{n-2}\ldots-na=2002$

2^2 vs 3^1

2^3 vs 3^2 (−)

2^4 vs 3^3 (−)

7^2 vs 8^1

7^3 vs 8^2

7^4 vs 8^3

7^8 vs 8^7

$a\cdot\left[a^n-a^{n-1}-na^{n-2}-\binom{n}{2}a^{n-3}-\cdots-n\right]=2002$

$8\log7$ vs $7\log8.$

$a=7:$ $49\mid n\cdot7+2002$

286

$7\mid n+6$

$n=1,$ $n=8.$ $n=15$

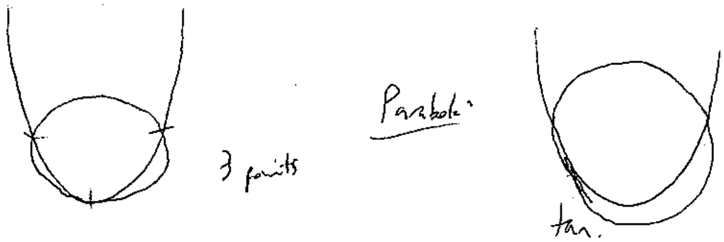
7^8 vs 8^7

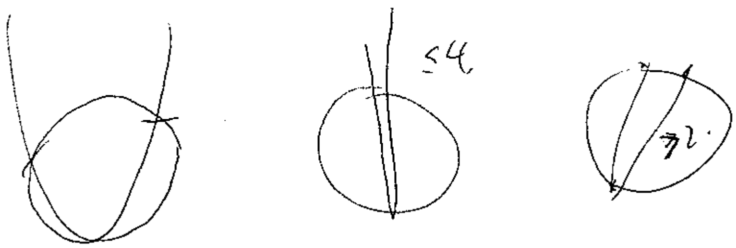
7^{15} vs 8^{14}

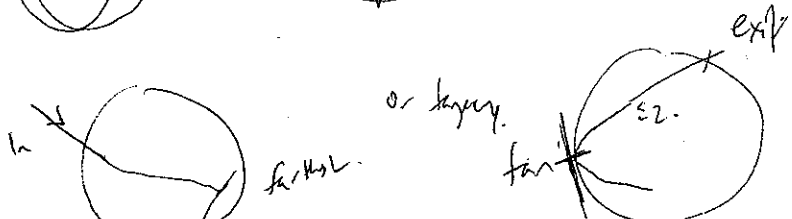
Mods

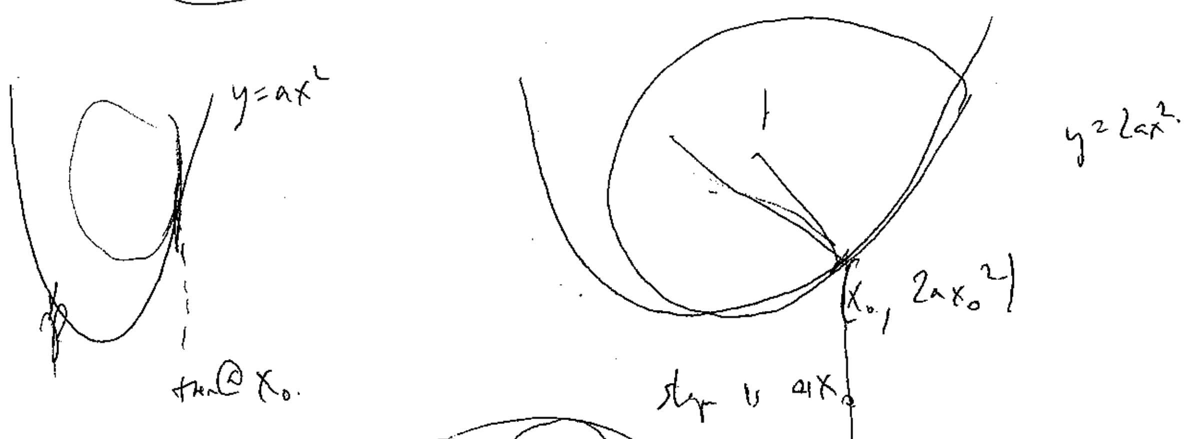
2001/A3

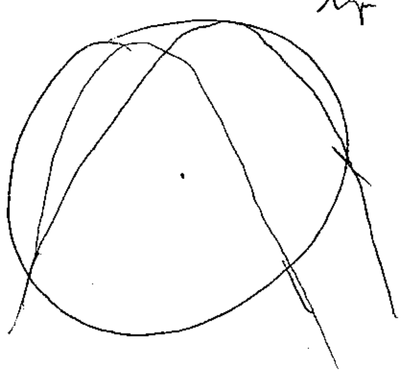
2011-10-09 ©











2011-10-09 ①

2001/A5

mod 3: $a^{n+1} - (a+1)^n \equiv 0 \pmod 3$ $\qquad a \mid 2\times 7\times 11\times 13.$

if n odd: square $- (a+1)^n \equiv 0 \pmod 3$

0, 1.

so either $(a+1)$ is mult. of 3 (impossible)

and a

or $(a+1)^n \equiv 1 \pmod 3$ and $a \not\equiv 0 \pmod 3$

$a+1 \equiv 1 \pmod 3$ since remove the square.

$a \equiv 0 \pmod 3$ again ✗

So n even $a^{odd} - \{0,1\} \equiv 0 \pmod 3$ $\quad a \equiv 1 \pmod 3$

$a^{odd} - (a+1)^{even} = 2001.$ $\qquad$ mod 2. ✗

if $2 \mid a$. Then $2^{odd} - odd^{even} = 2001$

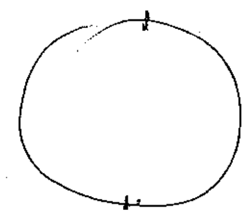
if $2 \mid a+1$. Then $4 \mid (a+1)^{even}$. mod 4: $a^{odd} \equiv 1 \pmod 4$

but $a^{even} \equiv \cancel{0}, 1 \pmod 4$

so $a \equiv 1 \pmod 4$ too.

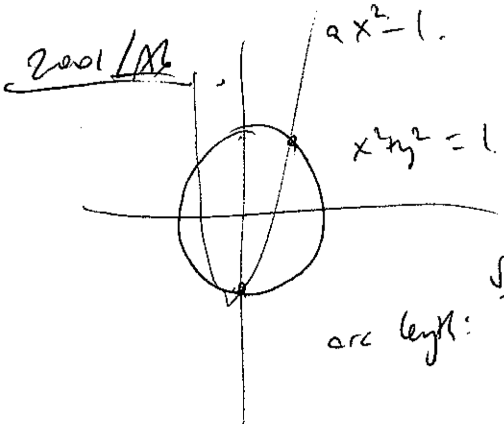
mod 4: $0 - \{0,1\} \equiv 1 \pmod 4$. ✗

So $a \equiv 1 \pmod 4$. $\qquad$ Choices: 2, 7, 11, 13 ✓, ~~77~~, 143, 91, ~~1001~~.



deriv of $\sqrt{t}$ at t is $\frac{1}{2\sqrt{t}}$.

2001/A6



$ax^2 - 1.$

$x^2+y^2=1.$

$x^2 + (ax^2-1)^2 = 1.$

$\cancel{x^2(1+a^2)+}$ $\qquad a^2x^4 + x^2(1-2a) \cancel{+1} = \cancel{1}.$

$a^2x^2 = 2a-1.$

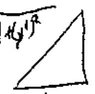
$x = \pm\sqrt{\frac{2a-1}{a^2}}$

$= \frac{\pm\sqrt{2a-1}}{a}.$

arc length: $\int_0^{\frac{\sqrt{2a-1}}{a}} dx\sqrt{1+4a^2x^2}$ vs. 2 ?

$y = ax^2-1.$

$y' = 2ax$



$dx\sqrt{1+(y')^2}$ $\quad dx\cdot y'$ $\quad dx$

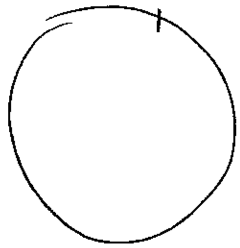
Scale out a. $= \int_0^{\frac{1}{\sqrt a}\sqrt{2-\frac1a}} 2a\sqrt{x^2+\frac{1}{4a^2}}\,dx. \approx \int_0^{\frac{1}{\sqrt a}\sqrt{2-\frac1a}} 2a\left(x + \frac{1}{4a^2}\frac{1}{2x}\right)dx.$

$\simeq (ax^2)^{\frac{1}{\sqrt a}\sqrt{2-\frac1a}} + \frac{1}{4a}\ln\left(\frac{1}{\sqrt a}\sqrt{2-\frac1a}\right)$

$= a\cdot\frac{2a-1}{a^2} = 2-\frac1a \qquad \frac{1}{4a}\ln\frac{1}{\sqrt a}$

2001/A6 2011-10-09 (E)

$\frac{1}{\sqrt a}\sqrt{2-\frac{1}{a}} \approx \left(\frac{2}{\sqrt a}, 1-\frac{2}{a}\right)$ $\sqrt{1-x^2} \approx 1-\frac{1}{2}x^2.$



$$\int_0^{\frac{1}{\sqrt a}\sqrt{2-\frac{1}{a}}} dx\left(1+\frac{1}{2}(2ax)^2\right) = \int_0^{\frac{1}{\sqrt a}\sqrt{2-\frac{1}{a}}} (1+2a^2x^2)\,dx.$$

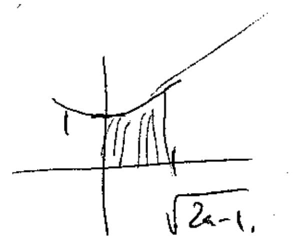
$$= \frac{1}{\sqrt a}\sqrt{2-\frac{1}{a}} + \frac{2}{3}a^2\left(\frac{1}{\sqrt a}\sqrt{2-\frac{1}{a}}\right)^3.$$

$$= \text{small} + \frac{2}{3}a^2\frac{1}{a^{3/2}}2\sqrt{2}. \to \infty ??$$

$$\int_0^{\frac{1}{\sqrt a}\sqrt{2-\frac{1}{a}}} 2ax\sqrt{1+\frac{1}{4a^2x^2}}\,dx = \int_0^{\sqrt a\sqrt{2-\frac{1}{a}}} 2y\sqrt{1-\frac{1}{(2y)^2}}\cdot\frac{dy}{a}$$

$y = ax.$
$dy = a\,dx.$

or just $\int_0^{\sqrt{2a-1}} \sqrt{1+(2y)^2}\,\frac{1}{a}\,dy.$



Now it's going far

$$\int_0^{\sqrt{2a-1}} \sqrt{1+(2y)^2}\,dy. > \int_0^{\sqrt{2a-1}} 2y\,dy = 2a-1.$$

$$\sqrt{1+(2y)^2}-2y = \sqrt{1+(2y)^2}-\sqrt{(2y)^2}$$

$$= \frac{1}{\sqrt{1+(2y)^2}+\sqrt{(2y)^2}}$$

$$> \frac{1}{3\times 2y}. = \frac{1}{6y}.$$

And we take $\int_1^{\sqrt{2a-1}} \frac{1}{6y}\,dy.$

$= \frac{1}{6}\lg\sqrt{2a-1}, \ldots$

overtakes (-1).

~~Now do~~

2011-10-09
F

30 10 segs

0.5, 1, 1.5, 2, 2.5

$\frac{1}{2}, \frac{3}{2}, \dots, \frac{10}{2}$.

1, 2, 3, 4, .., 10.

female: #7.

79/90

$4\frac{1}{2}$ = 9 segments ~~before~~ not include num

1 − P[her 7 entirely in the 9]

orders: 7, any

or 1, 7, any

or 2, 7, any.

$\frac{1}{10}$

$\frac{2}{10} \times \frac{1}{9} = \frac{2}{90}$

$\frac{9}{90}$ = ~~98~~ $\frac{11}{90}$

29

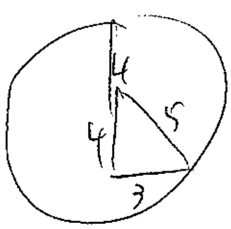
~~$(x^2-5x+8)(x^2$~~

$x^2 - 5x = -7$.

$(x^2-5x)^2 - 9$

40

28



~~72π~~

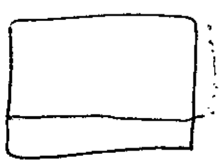
8×9 72π inside cyl.

$\frac{4}{3}\pi \times 125$ sphere

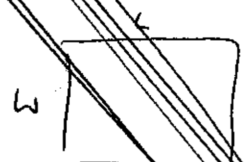
$\frac{500\pi}{3} - 72\pi$.

500
216
~~284π~~
3

27



+2.



~~LW~~ + 2 = (L+1)(W−2)

~~LW~~ − 2L + W − 2.

99 = 9 × 11.

97 = prime X.

98 = 7 × 14.

96 =

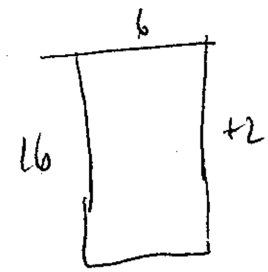
LW < 100

L = 7 W = 20 large

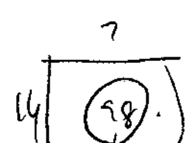
L = 6 W = 16 = 96.

4 = W − 2L

W = 2L + 4.



+2 = 98.



2011-12-11 (A)

Warm-Up

Min $x+\frac{4}{x}$ is $\frac{x+\frac{4}{x}}{2} \geq \sqrt{x\cdot\frac{4}{x}} = 2$

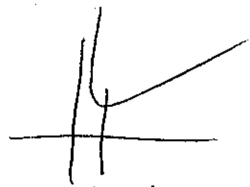
(equality)

so ≥ 4 always and can be achieved by $x=2$.

Min $(x^2+3) + \frac{4}{x^2+3}$ ≥ 2 again, but not so good

$3 \sim \infty$

$t+\frac{4}{t}$:



min $t=2$

increasing to right

So want to use $x=0$.

der: $1 - \frac{4}{t^2}$. for $t>2$, ~~neg~~

US 1997/4. $\exists$ reals sum ≥ 3. Suppose all ≤ 2.
Ssqr ≥ 9

2, 2, 2. ok

4 reals: 2, 2, 2, 2 $\geq$ 4 ok
4 4 4 4 = 16 ok, close

2, 2, 2, 2, 2
4, 4, 4, 4, 4 → 20 < 25.

Try $x \mapsto x-2$ Now want to be ≥ 0.

$x_1 + x_2 + \cdots + x_n \geq n - 2n = -n$

$(x_1+2)^2 + (x_2+2)^2 + \cdots + (x_n+2)^2 \geq n^2$

Maintain sum. Push apart → Ssqr increases

If $\leq 2-\epsilon$ for any ϵ,

Consider all $x_i \leq 2-\epsilon$
and sum $\geq n$
maximize Ssqr
→ get $f(\epsilon)$

If always $f(\epsilon) < n^2$
Then any given sequence will do this
Since n finite

$\overleftarrow{\;1 \quad 1\;}\!\longrightarrow$ eventually, just

Need sum $\geq n$.

$2-\epsilon$
$\vdots$
$n-1$ many

gets $(2-\epsilon)(n-1)$

$4(2-\epsilon)^2$ vs 44.

may as well make it $n - (2-\epsilon)(n-1)$
$= (\epsilon-1)n + (2-\epsilon)$.
$\cong 2-n + (n-1)\epsilon$

$n=4$: $4\epsilon - 4 + 2 - \epsilon$
$= -2 + 3\epsilon$
better at $2-\epsilon$.

worse than $2-n+\epsilon$

Ssqr is then:

$(\epsilon-1)^2 n^2 + \underbrace{2(\epsilon-1)n(2-\epsilon) + \underbrace{(2-\epsilon)^2 + (2-\epsilon)^2(n-1)}_{(2-\epsilon)^2 n}}$

$n^2[1-2\epsilon+\epsilon^2]$

$(2-\epsilon)n\cdot[(2-\epsilon) + 2(\epsilon-1)] = \epsilon\cdot(2-\epsilon)n$

$2-\epsilon+2\epsilon-2$.

2011-10-11 (B)

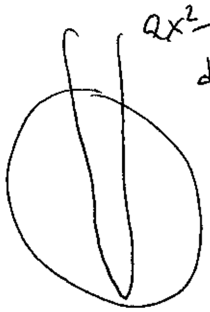
So: $\cdot$ $-(n-2-\epsilon)$ $> n-2$ $\quad$ $(n-1)$ $2-\epsilon$ $< 2.$

$$ssqr < (n-2-\epsilon)^2 + (n-1)(2-\epsilon)^2$$
$$= \underbrace{(n-2)^2}_{n^2-4n+4} - 2\epsilon(n-2) + \epsilon^2 + \underbrace{(n-1)}_{4n-4}[4-4\epsilon+\epsilon^2]$$

So $ssqr < (n-2)^2 + (n-1)\,2^2$ $= n^2.$ $\cancel{\epsilon}$. $\quad = n^2 - 2\epsilon(n-2) - 4\epsilon(n-1) + n\epsilon^2.$

So $\forall \epsilon$, the solution $f(\epsilon)$ is $< n^2$.
or just smooth to biggest guy. is the set same argument

2001/B6



ax^2-1
der: $2ax$
$x^2+y^2=1.$

intersection: $(ax^2-1)^2 + x^2 = 1$
$$a^2x^4 - 2ax^2 + x^2 = 0 \qquad x=0.$$
$$a^2x^2 = 2a-1$$
$$x = \pm\frac{\sqrt{2a-1}}{a}$$

$\frac{1}{2}$. arc-length $\displaystyle\int_0^{\frac{\sqrt{2a-1}}{a}} dx\,\sqrt{1+(2ax)^2}.$

big a. Scale: $y = ax$
$dy = a\,dx$

$$= \int_0^{\sqrt{2a-1}} \frac{dy}{a}\sqrt{1+4y^2}.$$

If it were $\displaystyle\int_0^{\sqrt{2a-1}} \frac{dy}{a}2y$, then we'd get $\frac{1}{a}\cdot(2a-1) = \frac{2a-1}{a} \approx 2$

But it's not:
Diff is: $\sqrt{1+4y^2} - 2y = \dfrac{1}{\sqrt{1+4y^2}+\sqrt{4y^2}} > \dfrac{1}{6y}$ for y big enough, say beyond 10.

Now we want correction $\displaystyle\int_{10}^{\sqrt{2a-1}} \frac{1}{6y}\frac{1}{a}\,dy = \frac{1}{6a}\left[\ln\sqrt{2a-1} - \ln 10\right]$

So, the arc length $> \frac{1}{a}\left[2a-1 + \frac{1}{3}(\ln\sqrt{2a-1} - \ln 10)\right]$

beats any constant eventually > 0

2011-06-11 (5)

Rearrangement Proof

How to get bigger.

If we have out of order: $\overset{a}{x_5} \le \overset{b}{x_9}$

but $\overset{c}{y_{\sigma(5)}} > \overset{d}{y_{\sigma(9)}}$

AM-GM 3 vars.

$(x+y+z)^3$ vs $27xyz$

$x^3+y^3+z^3+3(x^2y+\dots)+6xyz$ vs $27xyz$

21 xyz

Try τ which keeps σ, but swaps these.

$$ac + bd \overset{\le}{\text{vs}} ad + bc.$$

So gets bigger

$$a(c-d) \overset{\le}{\text{vs}} b(c-d)$$

$$a \overset{\le}{\text{vs}} b.$$

How about other direction?

Consider $z_i = -y_i$. Then rearrangement: $x_1 z_{\sigma(1)} + x_2 z_{\sigma(2)} + \dots + x_n z_{\sigma(n)} \le x_1 z_n + x_2 z_{n-1} + \dots + x_n z_1$

$$-x_1 y_{\sigma(1)} - x_2 y_{\sigma(2)} - \dots - x_n y_{\sigma(n)} \le -x_1 y_n - \dots$$

Cauchy-Schwarz

Want: $\dots (x_1 y_1)(x_5 y_5) \dots$ vs $x_1^2 y_5^2 = (x_1 y_5)(x_1 y_5)$

$x_1y_1 \quad x_1y_2 \quad x_1y_3 \dots x_1y_n \quad x_2y_1 \quad x_2y_2 \dots x_2y_n \dots$

$x_2y_1 \quad x_3y_1 \quad x_ny_1 \quad x_ny_2$

$(x_1y_1)(x_2y_2) \quad (x_1y_1)(x_ny_n) \quad (x_2y_2)(x_ny_n)$

$x_1y_2 \quad x_1y_3$ — same order

$x_1y_2 \quad x_1y_3$ all $x_i y_j$.

$\le$.

or $\log(\text{avg}) \ge \text{avg}(\log)$

$= \frac{1}{n}(\log x_1 + \log x_2 \dots)$

$\log \text{GM}$

AM-GM via rearrangement

reciprocals flip order.

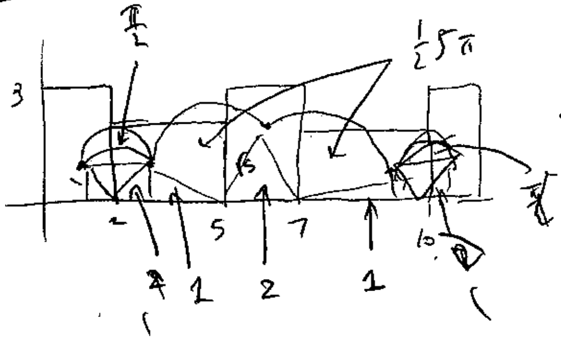
$$\frac{x_1}{G} \quad \frac{x_1x_2}{G^2} \quad \frac{x_1x_2x_3}{G^3} \dots \frac{x_1x_2\cdots x_n}{G^n} = 1$$

$$G \quad \frac{G^2}{x_1} \quad \frac{G^3}{x_1x_2} \quad \frac{G^4}{x_1x_2x_3} \quad \frac{G^n}{x_1x_2\cdots x_{n-1}} = G \times \left(1, \frac{G}{x_1}, \frac{G^2}{x_1x_2}, \dots \frac{G^{n-1}}{x_1\cdots x_{n-1}}\right)$$

$$x_1 + x_2 + \dots + x_n \ge G \cdot \left(\frac{x_1}{G}\cdot\frac{G}{x_1} + \frac{x_1x_2}{G^2}\cdot\frac{G^2}{x_1x_2} + \dots\right) = G \cdot n. \checkmark$$

2011-10-12 (A)

1991/A1



So ~~$3\pi + 6$~~ ~~$\frac{7}{2}\pi + 8$~~

$\frac{7}{2}\pi + 6$

$\frac{1}{2} \cdot 2\pi = \pi$.

1992/A2 $A^3 = B^3$. $A^2 + B^2$ invertible?

$A^2 B = B^2 A$.

I?

$A^3 B = A B^2 A$.

Ah, they're different

$B^4 = A B^2 A$.

Let A have ~~eigenvectors~~ eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, eigenvectors $v_1, v_2, \ldots, v_n$

So A^3 has $\lambda_1^3, \lambda_2^3, \ldots \lambda_n^3$

Same as B^3.

But $B^3 v_1 = \lambda_1^3 v_1$ so v_1 is eigenvector of B^3.

$B^3 = P^{-1} C^3 P$. So $P^{-1} C^3 P v_1 = \lambda_1^3 v_1$.

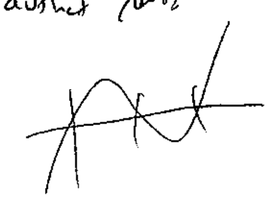
$C^3 P v_1 = \lambda_1^3 P v_1$

$$(A^2+B^2)(B-A) = A^2B + \underbrace{B^3 - A^3}_{0} - B^2A = 0.$$

So if A^2+B^2 invertible, then $B - A = 0$. $\Rightarrow A = B$ ↯

1992/A3 True for all quadratics with distinct roots

Cubic and higher? No way.



$(x-r_1)(x-r_2)\cdots(x-r_n)$

$\left(\frac{r_2-r_1}{2}\right)\left(-\frac{r_2-r_1}{2}\right)\left(\frac{r_1+r_2}{2}-r_3\right)\cdots\left(\frac{r_1+r_2}{2}-r_n\right)$

Does $P(x)\left[\frac{1}{x-r_1} + \frac{1}{x-r_2} + \cdots + \frac{1}{x-r_n}\right]$

Answer. $\frac{2}{r_2-r_1} - \frac{2}{r_2-r_1} + \frac{1}{\Theta} + \frac{1}{\Theta} + \cdots + \frac{1}{\Theta}$.

0. not 0

2011-10-16 Ⓐ

<u>2001/B4</u> $x - \frac{1}{x}$.

$\frac{2}{1} \to \frac{3}{2} \to \frac{3}{2} - \frac{2}{3} \to \frac{5}{6} - \frac{6}{5}$
$= \frac{5}{6}$ $\quad = \frac{-11}{30}$ always getting bigger.

$\frac{r}{s} - \frac{s}{r} = \frac{r^2 - s^2}{rs}$
(reduced) Can it have common factors? Then r with $r^2 - s^2 \Rightarrow r$ with s^2 ✗
So each go makes denominator strictly bigger.

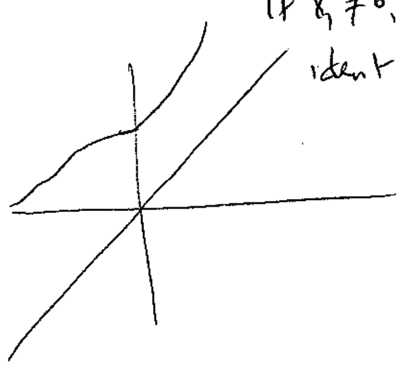
<u>2001/B5</u>

$g(g(x)) = a\, g(x) + bx$. $\qquad g(g(0)) = a\, g(0)$.

If cx, then $c^2 x = acx + bx$
$c^2 = ac + b$. $\quad c = \frac{a \pm \sqrt{a^2 + 4b}}{2}$ could reach $\frac{\frac{1}{2} + \sqrt{\frac{1}{4} + 2}}{2} = \frac{\frac{1}{2} + \frac{3}{2}}{2}$
$c^2 - ac - b = 0$. $\qquad = 1$.

Is it ever 0? then $g(x_0) = 0 \Rightarrow g(0) = 0 + b x_0 = b x_0$

Say $g(x_1) = x_1$. Then $x_1 = a x_1 + b x_1$ So any fixed point is at 0
If $x_1 \neq 0$, then $1 = a + b$ ✗.



ident So g is permanently on one side, to right.
and on other side, to left.

If $g(0) > 0$, then permanently above
So $g(x) > x$ always
$g(g(x)) > g(x)$,
$\Rightarrow g(x) < a\, g(x) + bx$.
$(1-a)\, g(x) < bx$.
$g(x) < \frac{b}{1-a} x$. $\qquad \frac{b}{1-a}$ is < 1
$\leq \alpha x$ where $\alpha < 1$. ✗. use big x

And if $g(x) < x$ always, then
$g(x) > a\, g(x) + bx$.
$(1-a)\, g(x) > bx$.
$g(x) > \frac{b}{1-a} x$. Use big negative x.

2011-10-16 (f)

2001/A5

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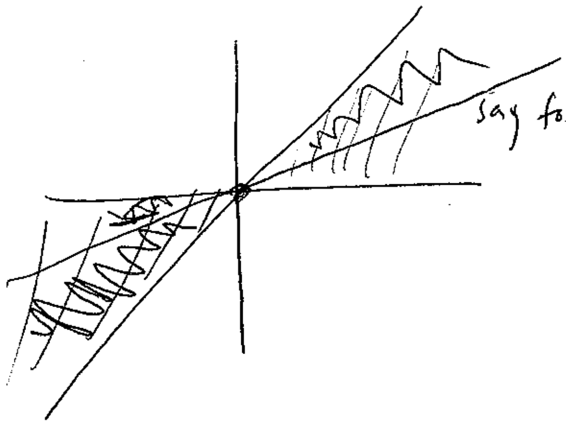
So $g(0) = 0$.

But now it must decide.



say for $x > 0$: $g(x) > x$.

Then $g(g(x)) > g(x)$ and for positive x,

$g(x) < a\,g(x) + bx$

$g(x) < \frac{b}{1-a} x$, problem

So for $x > 0$ in .

Similarly, if for $x<0$, $g(x) < x$, then $g(x) > a\,g(x) + bx$. ✱.

~~Compare~~ When ~~x~~ is $g(x) = \frac{1}{2}x$?

$\lambda(x) = \frac{g(x)}{x}$

$g(\frac{1}{2}x) = a \frac{1}{2}x + bx$.

$\lambda(g(x)) = a\lambda(x) + b$. $g(g(x)) = a\,g(x) + bx$.

injective: if $g(x) = g(y)$ then $bx = by$ ✓.

surjective? Say $0 \le g(x) < x$ for positive x.

Then $g(x) > a\,g(x) + bx$.

$g(x) > \frac{b}{1-a} x$

Sequence x, $g(x)$, $g(g(x))$, $g(g(g(x)))$ ...

or $\begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} \begin{bmatrix} g_{n-1} \\ g_n \end{bmatrix} = \begin{bmatrix} g_n \\ g_{n+1} \end{bmatrix}$

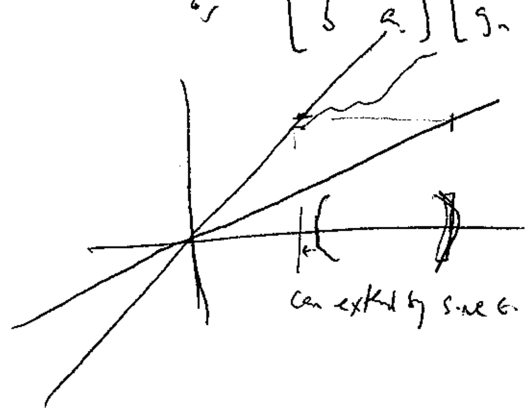
$g(g(g(x))) = a\,g(g(x)) + b\,g(x)$

$= a(a\,g(x) + bx) + b\,g(x)$.

$= [a^2 + b]\,g(x) + ab\,x$.

$g^4(x) = g(g^3 x) = a\,g^3(x) + b\,g^2(x)$

$=$



Can extend by some ε.

~~a_4~~ ~~$= a$~~

$g_4 = a g_3 + b g_2$. $0 < a, b < \frac{1}{2}$.

$x^2 - ax - b = 0$

$x = \frac{a \pm \sqrt{a^2+4b}}{2}$. one is negative.

recursion.

Decreasing. $g_n = \alpha \left(\frac{a - \sqrt{a^2+4b}}{2}\right)^n + \beta\left(\frac{a + \sqrt{a^2+4b}}{2}\right)^n$

dominant at small. $(-)$

dominant at big. bigger modulus.

Can't cope the [illegible]

2011-10-16 (c)

Putnam D.8

Po-Shen Loh

16 October 2011

1 Problems

Putnam 2001/B4. Let S denote the set of rational numbers different from $\{-1, 0, 1\}$. Define $f : S \to S$ by $f(x) = x - 1/x$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)}$ denotes f composed with itself n times.

Putnam 2001/B5. Let a and b be real numbers in the interval $(0, 1/2)$, and let g be a continuous real-valued function such that $g(g(x)) = ag(x) + bx$ for all real x. Prove that $g(x) = cx$ for some constant c.

Putnam 2001/B6. Assume that $(a_n)_{n\geq 1}$ is an increasing sequence of positive real numbers such that $\lim a_n/n = 0$. Must there exist infinitely many positive integers n such that $a_{n-i} + a_{n+i} < 2a_n$ for $i = 1, 2, \ldots, n-1$?

r_1 (−) $\qquad$ r_2 (+)

Even/odd. $X_n = g(X_{n-1})$. ... $X_n = \alpha\left(\frac{a-\sqrt{a^2+4b}}{2}\right)^n + \beta\left(\frac{a+\sqrt{a^2+4b}}{2}\right)^n$.

smaller abs val $\qquad$ larger abs val, but still < 1

So get ordered pairs $(x, g(x))$: (x_n, x_{n+1}),

If $\alpha \neq 0$

$(\alpha r_1^n + \beta r_2^n, \alpha r_1^{n+1} + \beta r_2^{n+1})$,

opp. sign.

so ratio is $\dfrac{\beta r_2^{n+1} + \alpha r_1^{n+1}}{\beta r_2^n + \alpha r_1^n}$

$= \dfrac{r_2 + \frac{\alpha}{\beta}\left(\frac{r_1}{r_2}\right)^n r_1}{1 + \frac{\alpha}{\beta}\left(\frac{r_1}{r_2}\right)^n}$

$\approx \left[r_2 + \frac{\alpha}{\beta}\left(\frac{r_1}{r_2}\right)^n r_1\right]\left[1 - \frac{\alpha}{\beta}\left(\frac{r_1}{r_2}\right)^n\right]$,

$= r_2 + \frac{\alpha}{\beta}\left(\frac{r_1}{r_2}\right)^n (r_1 - 1)$.

more negative.

but sign flips.

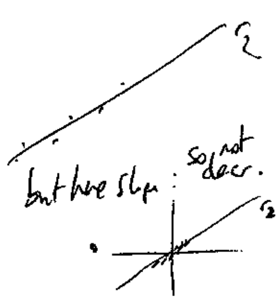
Invert? Then flipping?

Use (−) powers of n.

Now $X_{-n} = \alpha R_1^n + \beta R_2^n$.
(−)

Then $f(\ominus) = \oplus$ so
and $f(\oplus) = \ominus$ mono decr.

So if $\alpha \neq 0$, then mono decr. far away.
And if $\beta \neq 0$, then mono incr. Close in.
$\Rightarrow$ one must be 0



but here slope: so not decr.

1991/B1

2011-10-16 (D)

Reverse gap to nearest Square.

Largest product given Sum of 22 is Product of 2 with Sums up to 11.

Sum	~~1~~	2	3
Largest product	~~1~~	2	

Sum	0	1	2	3	4	5	6	7	8	9	10	11	12
Largest product	1	1	2	3	4	6	9	12 3·4	18 3.3.2	27	36	54	81

13	14	15	16	17	18	19	20	21	22
108	162	243	324	486	729 1458 ~~3^6~~				
$3^3\times4$	$3^4\times2$	3^5	3^4's/2's	$3^5\times2$	$3^6\times2$				

S
a

try all
0, S
1, S-1
2, S-2
3, S-3

3's and 2's.

(22) (20) × 2
or (19) × 3

$(20) = (18) \times 2$
or $(17) \times 3$ $= 3^6\times2$.

$3^{\cancel{5}}\times4$.

  13
 729
× 4
29.16

 36
× 12
 72
36
~~8~~32

$18^2 = 9^2\times2^2$.

$486 = 3\times162$
$= 3\times3^4\times2$
$= 3^5\times2$.

$108 = 12\times9$
$= 3^3\times4$

2011-10-17 (A)

1991/B1

Flip out square. To get exactly need to hit a square

— m — n^2 —

previous square

Since gap increases, need to be between successive

But gap is odd, so 5/3

1991/B2

$f(0) = f(0)^2 - g(0)^2$

$g(0) = 2f(0)g(0)$ If $g(0) \neq 0$ then $f(0) = \frac{1}{2}$ but $\frac{1}{2} = \frac{1}{4} - g(0)^2$ ✱

$g(0)[1-2f(0)] = 0$ So $g(0) = 0$

$f(0) = f(0)^2$ either 0 or 1

$g(0+y) = f(0)g(y) + 0$

$g(y) = f(0)g(y)$ $\forall y$, either $g \equiv 0$ or $\boxed{f(0)=1}$

noncomplete

$f(x+y) = f(x)f(y)$ $\forall x,y$.

$f(x+0) = f(x)f(0)$

$f(x) = f(x)f(0)$ either $f(x)$

$f(x)[1-f(0)] = 0$ either $\boxed{f(0)=1}$ or $f(x) \equiv 0$.

$f(x + -x) = f(x)f(-x) - g(x)g(-x)$

$1 = f(0)$ need symm need antisymm

$0 = g(x-x) = f(x)g(-x) + g(x)f(-x) \Rightarrow f(x)g(-x) = -f(-x)g(x)$

$f(0+x) = f(0)f(x) - g(0)g(x) = f(x)$.

$f'(x+y) = f'(x)f(y) - g'(x)g(y)$ $f'(0) = -g'(0)g(0)$ true.

$f'(0+y) = 0 - g'(0)g(y) \Rightarrow f'(y) = -g'(0)g(y)$

$g'(x+y) = f'(x)g(y) + g'(x)f(y) \Rightarrow g'(y) = g'(0)f(y) \rightarrow g'(0) = g'(0)\,1$ true

1991/B2

2011-10-17 (B)

$g(-x+y) = f(-x)\,g(y) + g(-x)\,f(y)$

$g(x+ -y) = f(x)\,g(-y) + g(x)\,f(-y)$

Also $f(x+y) = f(y+x)$ so ~~the~~ OK.

$f'(x)\,f(y) - g'(x)\,g(y) = f(x)\,f'(y) - g(x)\,g'(y)$

$0 = f'(x-x) = f'(x)\,f(-x) - g'(x)\,g(-x)$

$f'(x)\,f(-x) = g'(x)\,g(-x) = g'(0)\,f(x)\,g(-x)$

$-g(x)\,g'(0)\,f(-x)$

$g'(0+y) = g'(0)\,f(y)$

$g'(y) = g'(0)\,f(y)$

$g'(-x) = g'(0)\,f(-x)$

$f'(x+0) = -g(x)\,g'(0) = -g'(0)\,g(x)$

$g'(0) = g'(x-x) = f'(x)\,g(-x) + g'(x)\,f(-x)$

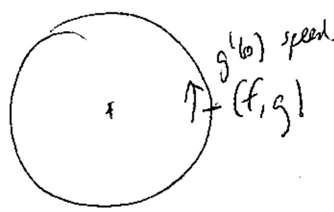
$f'(x) = -g'(0)\,g(x)$

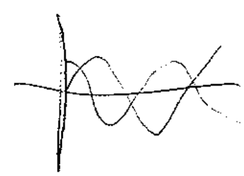
$g'(x) = g'(0)\,f(x)$

Initial value problem.

$f(0)=1 \quad g(0)=0$

$f'(0)=0 \quad g'(0)=c.$





1991/B3

Postage stamp Thm

Now: 4×6 ... $\}28$, 7×5 ... $\}28$ ⇒ can get 28 large.] → GCD is 4.

and 4×6 $\}20$, 5×7 $\}20$ ⇒ can get 20 large

→ GCD is 1. ✓

but 7×5 $\}35$ and 5×7 $\}35$ ⇒ gets 35 large