Luc Tartar
Remarks, Putnam preparation, Thursday November 3

VTRMC 2010/1, modified: Let A be a $n\times n$ integer matrix. Suppose

$$1+A+A^2+\cdots+A^{100}=0.$$

Show that $A^k+A^{k+1}+\cdots+A^{100}$ has determinant ±1 for every positive integer $k\leq 100$.
Solution: One uses the fact that, since $det(B_1B_2)=det(B_1)\,det(B_2)$, if B_1,B_2 are integer matrices such that $det(B_1B_2)\in\{-1,+1\}$, one deduces that $det(B_1),det(B_2)\in\{-1,+1\}$.

Multiplying $1+A+A^2+\cdots+A^{100}$ by $I-A$, one deduces that $I-A^{101}=0$, so that $det(A)^{101}=det(A^{101})=+1$, hence $det(A)=+1$.

Then, since $(I+A)\,(I+A^2+A^4+\cdots+A^{98}+A^{100})=I+A+A^2+\cdots+A^{100}+A^{101}=A^{101}$ has determinant $+1$, one deduces that $det(I+A)\in\{-1,+1\}$.

Then, since $(I+A+A^2)\,(I+A^3+A^6+\cdots+A^{96}+A^{99})=I+A+A^2+\cdots+A^{100}+A^{101}=A^{101}$ has determinant $+1$, one deduces that $det(I+A+A^2)\in\{-1,+1\}$.

Then, since $(I+A+A^2+A^3)\,(I+A^4+A^8+\cdots+A^{96}+A^{100})=I+A+A^2+\cdots+A^{102}+A^{103}=A^{101}+A^{102}+A^{103}=A^{101}(1+A+A^2)$ has determinant $\in\{-1,+1\}$, then $det(I+A+A^2+A^3)\in\{-1,+1\}$.

By induction, one deduces that for $1\leq j\leq 99$ one has $det(I+A+\cdots+A^j)\in\{-1,+1\}$, and this implies the desired result, since $det(A^k+A^{k+1}+\cdots+A^{100})=det^k(A)\,det(I+A+\cdots+A^j)$ for $j=100-k$.

Such an integer matrix A exists at least for n a multiple of 100, since the characteristic polynomial of

$$\text{the companion matrix}\ \begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 1\\ -c_0 & -c_1 & \dots & \dots & -c_{n-1}\end{pmatrix}\ \text{is}\ (-1)^n(c_0+c_1\lambda+\ldots+c_{n-1}\lambda^{n-1}+\lambda^n),$$

and by taking $n=100$ and all $c_j=1$ one obtains a desired integer matrix. I suppose that such an integer matrix mail only exist if n is a multiple of 100.

Putnam 2010/B6: Let A be an $n\times n$ real matrix. For each integer $k\geq 0$, let $A^{[k]}$ be the matrix obtained by raising each entry of A to the k-th power. Show that if $A^k=A^{[k]}$ for $k=1,2,\ldots,n+1$, then $A^k=A^{[k]}$ for all $k\geq 1$.
Hint: By linearity, the hypothesis implies that for every polynomial Q of degree $\leq n$ one has $\big(Q(A)\big)_{i,j}=Q(A_{i,j})$. By the theorem of Cayley–Hamilton, A satisfies $P_c(A)=0$, where P_c is the characteristic polynomial $P_c(\lambda)=det(\lambda\,I-A)$. One then has $P_c(A_{i,j})=0$ for all entries $A_{i,j}$. If P is any polynomial (like $P(x)=x^k$ for an arbitrary value of k) then the Euclidean division algorithm gives $P=P_cQ_1+Q_2$ with $degree(Q_2)\leq n-1$, and $P(A)=Q_2(A)$, and $P(A_{i,j})=Q_2(A_{i,j})$ for all entries, hence $\big(P(A)\big)_{i,j}=P(A_{i,j})$ for all polynomials P and all entries.

Notice that one does not need the hypothesis for $k=n+1$.

Of course, diagonal matrices have this property, but it is interesting to observe that there are non-diagonal ones as well: for $n=2$, since $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ implies $A^2=\begin{pmatrix} a^2+b\,c & b\,(a+d)\\ c\,(a+d) & d^2+b\,c\end{pmatrix}$, so that $A^2=\begin{pmatrix} a^2 & b^2\\ c^2 & d^2\end{pmatrix}$ means $b\,c=0$, $b^2=b\,(a+d)$, and $c^2=c\,(a+d)$, and choosing $b=0$ and $c=a+d$ gives an infinite family of non-diagonal solutions.

VTRMC 2009/5: Let A,B be 3×3 matrices with $B\neq 0$ and $A\,B=0$. Prove that there exists a 3×3 matrix D such that $A\,D=D\,A=0$.
[Of course, the question should ask that D be different from 0.]
Solution: Since the image of B is not restricted to $\{0\}$, the property of B is that its kernel is not restricted to $\{0\}$. Then the image of A is not $\mathbb{R}^3$ since the dimension of the kernel $ker(A)$ plus the dimension of the image $Im(A)$ is 3. One finds all solutions D by asking that D is 0 on $Im(A)$, and on a supplement X of $Im(A)$, one defines D to take values in $ker(A)$.

As I mentioned, using the Cayley–Hamilton theorem, i.e. $P_c(A) = 0$ for the characteristic polynomial P_c of A, does not always permit to take $D = Q(A)$ for the polynomial $Q(x) = \frac{P_c(x)}{x}$, since for a matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ one has $A^2 = 0$, and it that case it would give $D = 0$. One should then take $Q(x) = \frac{P_{min}(x)}{x}$ for the minimal polynomial P_{min}, which has the eigenvalue λ as a root with multiplicity equal to the geometric multiplicity of that eigenvalue, and not the algebraic multiplicity.