3. Pidgeonhole principal

Po-Shen Loh

CMU Putnam Seminar, Fall 2011

In fact, a pigeonhole is neither a part of a pigeon nor a container for pigeons, but rather a box used for filing papers (such as the faculty mailboxes that might line the walls of your school's main office). However, that fact is boring, so you may as well ignore it like everyone else does.

— GDC

1 Classical results

Ramsey's Theorem. For every positive integer t, there is a number n such that every red/blue coloring of the edges of K_n contains a monochromatic clique K_t, i.e., t vertices such that all $\binom{t}{2}$ edges between them are of the same color.

Hamming codes. A license plate has seven binary digits (0 or 1), and may have leading zeros. If two plates must always differ in at least three places, what is the largest number of plates that is possible?

2 Problems

GA 44. Inside a circle of radius 4 are chosen 61 points. Show that among them there are two at distance at most $\sqrt{2}$ from each other.

GA 46 (Moscow Math Olympiad). Show that any convex polyhedron has two faces with the same number of edges.

IMO 1993/6a. There are $n > 1$ lamps $L_0, L_1, \ldots, L_{n-1}$ in a circle. We use L_{n+k} to mean L_k. A lamp is at all times either on or off. Initially they are all on. Perform steps $s_0, s_1, \ldots$ as follows: at step s_i, if L_{i-1} is lit, then switch L_i from on to off or vice versa, otherwise do nothing. Show that there is a positive integer $M(n)$ such that after $M(n)$ steps all the lamps are on again;

USAMO 2000/4. Find the smallest positive integer n such that if n squares of a 1000×1000 chessboard are colored, then there will exist three colored squares whose centers form a right triangle with sides parallel to the edges of the board.

USAMO 1990/1. A license plate has six digits from 0 to 9 and may have leading zeros. If two plates must always differ in at least two places, what is the largest number of plates that is possible?

IMO 2011/2. Let S be a finite set of at least two points in the plane. Assume that no three points of S are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in S$. The line rotates clockwise about the pivot P until the first time that the line meets some other point Q belonging to S. This point Q takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of S. This process continues indefinitely. Show that we can choose a point P in S and a line ℓ going through P such that the resulting windmill uses each point of S as a pivot infinitely many times.