2. Induction

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1 Classical results

1. Prove that for every positive integer n, there exists a finite set of points in the plane such that for every point of the set there exist exactly n other points of the set at distance equal to 1 from that point.

2. A *Hadamard Matrix* is an $n \times n$ square matrix, all of whose entries are $+1$ or -1, such that every pair of distinct rows is orthogonal. In other words, if the rows are considered to be vectors of length n, then the dot product between any two distinct row-vectors is zero. Show that infinitely many Hadamard Matrices exist.

3. Hadamard Conjecture (open): for every positive integer k, there is a Hadamard Matrix of order $4k$. The first unknown case is $4k = 668$.

2 Problems

GA 22. Prove that for any positive integer $n \geq 2$ there is a positive integer m that can be written simultaneously as a sum of 2, 3, ..., n squares of nonzero integers.

USAMO 2003/1. Prove that for every positive integer n, there exists an n-digit number divisible by 5^n, all of whose digits are odd.

USAMO 1997/4. An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver matrix* if, for each $i = 1, 2, \ldots, n$, the i-th row and the i-th column together contain all elements of S. Show that:

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

GA 18. Prove that for any $n \geq 1$, a $2^n \times 2^n$ checkerboard with any 1×1 square removed can be tiled by L-shaped triominoes.

3 Unrelated bonus problem

Z. Feng 1997. For all real $a > 0$, prove that

$$\sqrt{a + \sqrt{2a + \sqrt{3a + \sqrt{4a + \sqrt{5a}}}}} < \sqrt{a} + 1\,.$$