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Remarks, Putnam preparation, Thursday September 8

2: Besides having entries ± 1, the transposed matrix $M = H^T$ must have orthogonal columns, so that $(M\,e_i, M\,e_j) = 0$ for $i \neq j$, i.e. $M^T M$ should be diagonal. For $n = 2$, one takes $M_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ for example, so that $M_2^T M_2 = 2I$. Assuming that M_n is an $n \times n$ matrix such that $M_n^T M_n = D_n$ is diagonal, then the $2n \times 2n$ matrix written in $n \times n$ blocks as $M_{2n} = \begin{pmatrix} M_n & -M_n \\ M_n & M_n \end{pmatrix}$ satisfies $M_{2n}^T M_{2n} = \begin{pmatrix} 2D_n & 0 \\ 0 & 2D_n \end{pmatrix}$, which is diagonal. Starting from M_2 one then generates transposed of Hadamard matrices whose size is any power of 2.

GA 22: Since 5^2 is a square which can be written as a sum of 2 squares, because $5^2 = 3^2 + 4^2$, one deduces that 5^{2m} is a square which can be written as a sum of j squares for $j = 2, \ldots, 2^m$ for $m \geq 1$.

Indeed, if $5^{2n} = a_1^2 + \ldots + a_j^2$, then $5^{2n+2} = 5^2 a_1^2 + \ldots + 5^2 a_j^2$ and replacing some 5^2 by $3^2 + 4^2$ one can write 5^{2n+2} as a sum of k squares for $k = j, j+1, \ldots, 2j$.

USAMO 1997/4: Such a matrix M does not exist if n is odd ≥ 3. There are at least $n-1$ integers which do not appear in the diagonal, and let a be one of them; if $a = M_{i,j}$ with necessarily $i \neq j$, then a cannot appear a second time in rows or columns of index i or j, and for a different index k, a must appear in row or column k, so that each time a appears it selects a pair in $\{1, \ldots, n\}$ and the union of these pairs must be $\{1, \ldots, n\}$, so that n must be even.

It seems to me that the following construction holds for n being any power of 2. It is not important that the integers used be $\{1, \ldots, 2n-1\}$ and one can use $\{-n+1, \ldots, n-1\}$, and one first constructs a symmetric matrix A_n with 0 in the diagonal, and positive entries outside the diagonal and such that each row contains all the integers from 0 to $n-1$, then the desired matrix is obtained by multiplying all the elements below the diagonal by -1. If J_n denotes the $n \times n$ matrix with all entries 1, and A_n has the preceding property, then the $2n \times 2n$ matrix A_{2n} written in $n \times n$ blocks as $\begin{pmatrix} A_n & n\,J_n + A_n \\ n\,J_n + A_n & A_n \end{pmatrix}$ satisfies the same property for the double size. Starting with $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ permits then to solve the question for all powers of 2.

Z. Feng 1997: For $a > 0$, and $m \geq 1$, one defines $F_m(a) = \sqrt{a + \sqrt{4a + \sqrt{4^2 a + \ldots \sqrt{4^{m-1} a + \sqrt{4^m a}}}}}$, so that $F_m(a)^2 = a + F_{m-1}(4a)$, and one may define $F_0(a) = \sqrt{a}$, so that the formula holds for $m \geq 1$.

By induction, one proves that $F_m(a) < \sqrt{a} + 1$ for all $a > 0$, since it is true for $m = 0$, and if it is true for $m-1$, one deduces that $F_m(a)^2 = a + F_{m-1}(4a) < a + \sqrt{4a} + 1 = (\sqrt{a} + 1)^2$, so that $F_m(a) < \sqrt{a} + 1$.