Even more advanced Putnam training

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D-Day − 1

1 Problems

Putnam 2001/B4. Let S denote the set of rational numbers different from $\{-1, 0, 1\}$. Define $f : S \to S$ by $f(x) = x - \frac{1}{x}$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)}$ denotes f composed with itself n times.

Putnam 2008/A4. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ xf(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

Putnam 2007/B4. Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that

$$(P(x))^2 + (Q(x))^2 = x^{2n} + 1,$$

and $\deg(P) > \deg(Q)$.