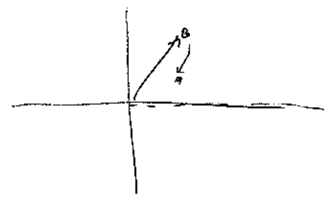
2005/A3.

P: all zeros abs val 1.



positive branch.

$$g(z) = \frac{p(z)}{z^{\frac{n}{2}}}.$$

$$g'(z) = \frac{z^{\frac{n}{2}} p'(z) - p(z)\frac{n}{2} z^{\frac{n}{2}-1}}{z^n}$$

$$= \frac{p'(z)}{z^{\frac{n}{2}}} - \frac{n}{2}\frac{p(z)}{z^{\frac{n}{2}}}\frac{1}{z}.$$

then: $z p'(z) = \frac{n}{2} p(z)$

$$z = \frac{n}{2}\frac{p(z)}{p'(z)}.$$

$$p(z) = (z-r_1)(z-r_2)\cdots(z-r_n)$$

$$\frac{p(z)}{\sqrt{z}^n} = \left(\sqrt{z} - \frac{r_1}{\sqrt{z}}\right)\left(\sqrt{z} - \frac{r_2}{\sqrt{z}}\right)\cdots\left(\sqrt{z} - \frac{r_n}{\sqrt{z}}\right) = \cancel{g(z)}\, g(z)$$

Der: $\displaystyle\sum_{i=1}^{n} \frac{g(z)}{\sqrt{z} - \frac{r_i}{\sqrt{z}}} \cdot \left(+\frac{1}{2\sqrt{z}} + \frac{r_i}{2z^{3/2}}\right)$

$$= g(z) \times \sum_{i=1}^{n} \frac{\frac{1}{2z^{3/2}}(z+r_i)}{\frac{1}{\sqrt{z}}(z-r_i)} = g(z)\sum_{i=1}^{n} \frac{1}{2z}\,\frac{z+r_i}{z-r_i}$$

zero iff when $g(z)=0$, or..

$\displaystyle\sum \frac{z+r_i}{z-r_i}$ when $|r_i| = 1$.

Show this is never 0?

No, but if $n=1$, it is only at $-r_1$.

What if $p(z) = z-1$.

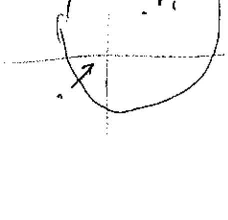
$$g(z) = \frac{z-1}{\sqrt{z}} = \sqrt{z} - \frac{1}{\sqrt{z}}$$

$$g'(z) = \frac{1}{2\sqrt{z}} + \frac{1}{2z^{3/2}} = \frac{1}{2z^{3/2}}(z+1)$$ $z=-1$ works.

$$\frac{1}{2i} - \frac{1}{2i} = 0. ✓$$

$$\frac{z+r_1}{z-r_1} + \frac{z+r_2}{z-r_2} = \frac{z^2 - r_2 z + r_1 z - r_1 r_2 + z^2 - r_1 z + r_2 z - r_1 r_2}{(z-r_1)(z-r_2)}$$

$$= \frac{2z^2 - 2r_1 r_2}{(z-r_1)(z-r_2)} \Rightarrow z^2 = r_1 r_2,$$ still modulus 1



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(2)

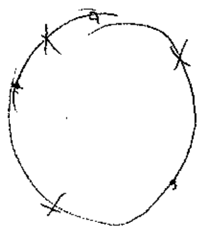
So $n + 2\left[\frac{r_1}{z-r_1} + \frac{r_2}{z-r_2} + \dots + \frac{r_n}{z-r_n}\right]$

$$\frac{r_1}{z-r_1} + \frac{r_2}{z-r_2} = \frac{z(r_1+r_2) - 2r_1r_2}{\Pi}$$

$$\frac{z+r_1}{z-r_1} + \frac{z+r_2}{z-r_2} + \frac{z+r_3}{z-r_3} = \frac{1}{\Pi}\left[3z^3 + z^2\begin{pmatrix} r_1 - r_2 - r_3 \\ -r_1+r_2-r_3 \\ -r_1-r_2+r_3\end{pmatrix} + z\begin{pmatrix} -r_1r_2 + r_2r_3 - r_3r_1 \\ -r_1r_2 - r_2r_3 + r_3r_1 \\ +r_1r_2 - r_2r_3 - r_3r_1\end{pmatrix} + 3r_1r_2r_3\right]$$

$$-z^2(r_1+r_2+r_3) - z(r_1r_2+r_2r_3+r_3r_1)$$

No.

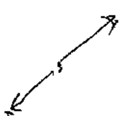


Re part:

$$\frac{(z+r_i)(\bar z - \bar r_i)}{|z-r_i|^2} = \frac{|z|^2 - z\bar r_i + \bar z r_i - |r_i|^2}{|z-r_i|^2}$$

$y - \bar y$:

$$= \frac{|z|^2 + 2\,\mathrm{Im}(\bar z r_i) - 1}{|z-r_i|^2}$$ Imag.



$$\frac{z_i+1}{z_i-1} = 1 + \frac{2}{z_i-1}$$

So to be 0, need $\sum \frac{\mathrm{Im}\, z r_i}{|z-r_i|^2} = 0$

or Real part: $\sum \frac{|z|^2-1}{|z-r_i|^2} = 0$

If $|z|=1$, done.

Else $= (|z|^2-1)\sum \frac{1}{|z-r_i|^2}$ always > 0

✓

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3

2006/B1

$-1 -1 -1 = 1.$

$x^3+3xy+y^3=1.$

$x=0: \ y=1.$

$x+y=1$ is root.

$x=1: \ y=0$

$x^3+3xy+y^3-1$ divided by $x+y-1$

$x=-1:$ no

$x=2: \quad 8+6y+y^3=1 \qquad y=-1$

$6y+y^3=-7.$

$$x^3+3xy+y^3-1$$

$$(x+y-1)(x^2+y^2+1) = x^3+xy^2+x+x^2y+y^3+y-x^2-y^2-1.$$

DIFF: $3xy-xy^2-x-x^2y-y+x^2+y^2$

~~$= xy(2-y-$~~

~~$(x+y-1)(x) = x^2+xy-x.$~~

~~$(x+y-1)(-x-y) = -x^2-2xy+y^2+x+y$~~

$$(x+y-1)(-xy) = -x^2y-xy^2+xy$$

DIFF: $2xy-x-y+x^2+y^2$

$$(x+y-1)(x+y) = x^2+2xy+y^2-x-y.$$

So it's $(x+y-1)\left[x^2+y^2+1-xy+x+y\right].$ $\quad x=y$ and $x,y=-1.$

ellipse

$$\tfrac12x^2-xy+\tfrac12y^2 + \tfrac12x^2+x+\tfrac12 + \tfrac12y^2+y+\tfrac12$$

$$\tfrac12(x-y)^2 + \tfrac12(x+1)^2 + \tfrac12(y+1)^2$$