General strategy

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1 Problems

Putnam 2010/A0. When and where is the Putnam?

Putnam 2006/B4. Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Let k be given, $0 \le k \le n$. Find the maximum, over all vector subspaces $V \subset \mathbb{R}^n$ of dimension k, of the number of points in $V \cap Z$.

Putnam 2006/A4. Let $S = \{1, \ldots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

(i) $\pi(k) > \pi(k+1)$ for $k = 1$;

(ii) $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1 < k < n$;

(iii) $\pi(k-1) < \pi(k)$ for $k = n$.

For example, if $n = 5$ and π takes values at $1, 2, 3, 4, 5$ of $2, 1, 4, 5, 3$, then π has a local maximum of 2 at $k = 1$, and a local maximum of 5 at $k = 4$. What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

Putnam 2008/B4. Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0)$, $h(1)$, $\ldots$, $h(p^2-1)$ are distinct modulo p^2. Show that $h(0)$, $\ldots$, $h(p^3-1)$ are distinct modulo p^3.

Putnam 2008/A4. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \le e \\ x f(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

Putnam 2007/B4. Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that

$$(P(x))^2 + (Q(x))^2 = x^{2n} + 1,$$

and $\deg(P) > \deg(Q)$.

Putnam 2007/A4. A *repunit* is a positive integer whose digits in base 10 are all ones. Find all polynomials f with real coefficients such that if n is a repunit, then so is $f(n)$.