2003/A1

2010-11-16

①

n $\begin{cases} k=1: & n. \\ k=2: & \\ \vdots & \\ k=n: & 1+1+1+1+\dots+1 \end{cases}$

Is there a way? Yes, there is one: dividing candy in school,
Go around, +1 to each.
Stop when run out.

Is there more than one way?

$$\underbrace{x \quad x \quad x \quad x \quad \overbrace{(x+1) \quad (x+1) \quad \dots \quad (x+1)}^{r}}_{k \text{ total.}}$$

$n = \text{Sum} \equiv r \pmod{k}$, so r is uniquely determined

$r = 0 \dots k-1$

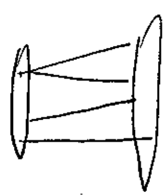
$n = kx + r$
solve for x.

2001/B1

$$\begin{matrix} 1 & 2 & 3 & 4 & \dots & n \\ n+1 & n+2 & & & & 2n \\ & \vdots & & & & \\ & & & & & n^2. \end{matrix}$$

Red is sum of permutation matrices.
→ constant sum.
One per row: gets $+0+n+2n+3n+\dots+(n-1)n$.
One per column: gets $+1+2+3+\dots+n$.
So each permutation matrix gets equal sum
Red squares is $\frac{n}{2}$ of them.
All squares is n of them

<u>TH</u>. (HALL). Bipartite graph.

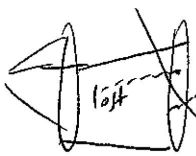


<u>DEF</u>. Perfect matching from $L \to R$: associate distinct choice on R with each from L.

→ Obviously need that every vtx on L has ≥ 1 nbr on R.
More generally, need that every $S \subseteq L$ has $|N(S)| \geq |S|$.

<u>TH</u>. (HALL) This is sufficient.

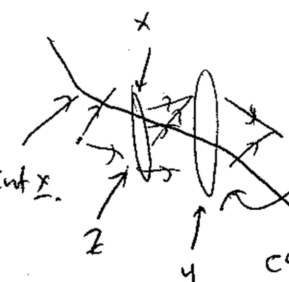
<u>PF</u>. Induction or max-flow-min-cut.



every vtx in $N(z)$ lost
loses by cut edge
whether or not it is
on same side of cut as z

←

cut x.



Smallest cut puts all non-$N(X)$ vtxs in Y.
$\leq R-x$ of them

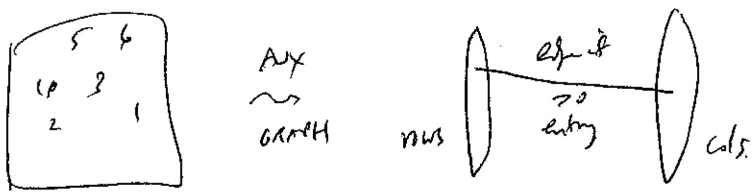
2010-11-16
②

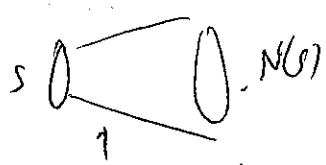
Th Every matrix with nonnegative integer entries, with all row sums = all col sums is sum of permutation matrices
exactly one 1 in each row/column.

$$\begin{pmatrix} & 1 & & 1 \\ 1 & & 1 & \\ & 1 & & \end{pmatrix}\begin{pmatrix}x_1\\x_2\\ \vdots \\ x_n\end{pmatrix} = \text{permute}$$

Pf. Suffices to find one

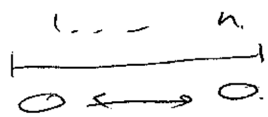
Let K be common sum





Sum on all edges is exactly SK
But each vtx in $N(S)$ can only "catch" up to K.
So $|N(S)| \geq \frac{SK}{K} = S$

2002/A3. Flip the set?



How many sets stay stationary? Only if perfect pairing.

If n even, then avg is $\frac{n+1}{2}$, not integer. So none → rest is even.

If n odd, then avg for all is $\frac{n+1}{2}$, integer.

For each of $\left(\frac{n+1}{2}\right)$, $\left(\frac{n-1}{2}, \frac{n+3}{2}\right)$, $\left(\frac{n-3}{2}, \frac{n+5}{2}\right)$, ..., $(1, n)$

we either take whole pair or none.

So have 2^{stuff} ways. ← That overcounts empty set.

⇒ odd # ways

total is odd

2010-11-16

3

2006/A4

~~X = average # of local maxima.~~

~~I_1 = average value.~~

Consider π. Ask: ~~is there local maximum at 1? If yes,~~ how many local maxima. Call answer $X(\pi)$.

Need to calculate avg value of $X(\pi)$ where π is uniformly random permutation.

Let $I_1(\pi) = \begin{cases} 1 & \text{if 1 is local maximum} \\ 0 & \text{else} \end{cases}$

Let $I_2(\pi) = \begin{cases} 1 & \text{---} \\ 0 & \end{cases}$

$\vdots$

$I_n(\pi)$.

Then $X(\pi) = I_1(\pi) + I_2(\pi) + \dots + I_n(\pi)$

avg $I_1(\pi) = ?$

random permutation. What is chance that it gives local max at 1?

↳ choose random value at 1, then random value from rest at 2, then ...

or, choose 2 values to be at $\{1,2\}$, then randomly pick which one is 1, then 2, then fill rest.

$\Rightarrow$ chance of local max at 1 is $\frac{1}{2}$.

avg $I_2(\pi)$ = choose what values for positions $\{1,2,3\}$

Then decide who will be on #2, then who will be #1, then who will be #3.

Will have local max if the #2 choice is biggest $\rightarrow \frac{1}{3}$.

$\Rightarrow$ Ans = $\frac{1}{2} + \frac{n-2}{3} + \frac{1}{2}$.

Turán. Random permutation. Take anybody that precedes all of his nbrs.

$I_v = \begin{cases} 1 & \text{if } v \text{ precedes nbrs} \\ 0 & \end{cases}$

$E[I_v]$ = expose positions that will be occupied by $v \cup N(v)$, but not who is where.

Then say who is where... v is at front $\frac{1}{d(v)+1}$ of the time

$= \frac{1}{d(v)+1}$

$\mathbb{E}[X] = \sum_v \frac{1}{d(v)+1} \geq \sum_v \frac{1}{(\text{avg deg})+1} = \frac{n}{d+1}$. $\Rightarrow$ has a choice.

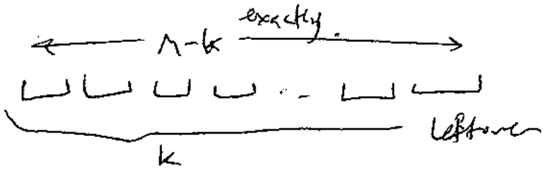
2005/B4

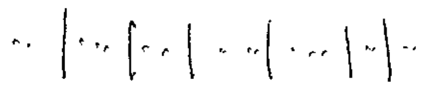
2010-11-16
(4)

k = # of nonzero → Split ≤ m mass into k bins, with no empty, then × 2^k for sign

k = 0, 1, …, n. ↑ $\binom{n}{k}$ ≤ m−k mass into k bins, possibly empty

name the bins




m−k dots

k bars

Any ordering works → $\binom{m}{k}$.

So $f(m,n) = \sum_{k=0}^{n} 2^k \binom{m}{k}\binom{n}{k}$ Symmetric

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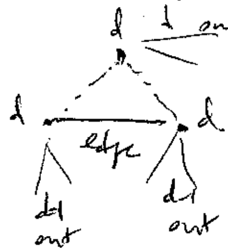
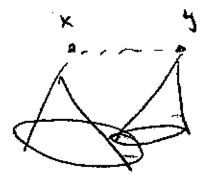
Turán (2) Independence is equivalence relation.

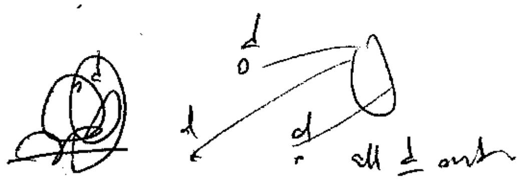
(1) Indep vtxs have same degree





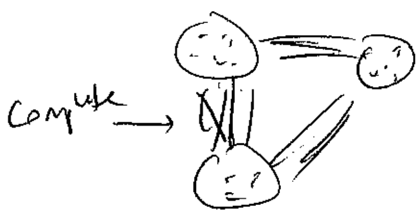
If d(x) < d(y).
Then clone y → x.
Can't make any new K_t since they would need to involve x, and that would be like y.

vs.



(+1) gain.

⇒ all indep sets are disjoint.



⇒ complete multipartite graph
⇒ fewer than ≤ t−1 parts
⇒ optimized by Turán graph