Solutions to the 64th William Lowell Putnam Mathematical Competition
Saturday, December 6, 2003

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A1 There are n such sums. More precisely, there is exactly one such sum with k terms for each of $k = 1, \dots, n$ (and clearly no others). To see this, note that if $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$, then

$$\begin{aligned} ka_1 &= a_1 + a_1 + \cdots + a_1 \\ &\leq n \leq a_1 + (a_1+1) + \cdots + (a_1+1) \\ &= ka_1 + k - 1. \end{aligned}$$

However, there is a unique integer a_1 satisfying these inequalities, namely $a_1 = \lfloor n/k \rfloor$. Moreover, once a_1 is fixed, there are k different possibilities for the sum $a_1 + a_2 + \cdots + a_k$: if i is the last integer such that $a_i = a_1$, then the sum equals $ka_1 + (i-1)$. The possible values of i are $1, \dots, k$, and exactly one of these sums comes out equal to n, proving our claim.

Note: In summary, there is a unique partition of n with k terms that is "as equally spaced as possible". One can also obtain essentially the same construction inductively: except for the all-ones sum, each partition of n is obtained by "augmenting" a unique partition of $n-1$.

A2 **First solution:** Assume without loss of generality that $a_i + b_i > 0$ for each i (otherwise both sides of the desired inequality are zero). Then the AM-GM inequality gives

$$\begin{aligned} &\left(\frac{a_1 \cdots a_n}{(a_1+b_1)\cdots(a_n+b_n)} \right)^{1/n} \\ &\leq \frac{1}{n} \left(\frac{a_1}{a_1+b_1} + \cdots + \frac{a_n}{a_n+b_n} \right), \end{aligned}$$

and likewise with the roles of a and b reversed. Adding these two inequalities and clearing denominators yields the desired result.

Second solution: Write the desired inequality in the form

$$(a_1+b_1)\cdots(a_n+b_n) \geq [(a_1\cdots a_n)^{1/n} + (b_1 \cdots b_n)^{1/n}]^n,$$

expand both sides, and compare the terms on both sides in which k of the terms are among the a_i. On the left, one has the product of each k-element subset of $\{1, \dots, n\}$; on the right, one has $\binom{n}{k}(a_1\cdots a_n)^{k/n} \cdots (b_1 \dots b_n)^{(n-k)/n}$, which is precisely $\binom{n}{k}$ times the geometric mean of the terms on the left. Thus AM-GM shows that the terms under consideration on the left exceed those on the right; adding these inequalities over all k yields the desired result.

Third solution: Since both sides are continuous in each a_i, it is sufficient to prove the claim with $a_1, \dots, a_n$ all positive (the general case follows by taking limits as some of the a_i tend to zero). Put $r_i = b_i/a_i$; then the given inequality is equivalent to

$$(1+r_1)^{1/n} \cdots (1+r_n)^{1/n} \geq 1 + (r_1 \cdots r_n)^{1/n}.$$

In terms of the function

$$f(x) = \log(1+e^x)$$

and the quantities $s_i = \log r_i$, we can rewrite the desired inequality as

$$\frac{1}{n}(f(s_1) + \cdots + f(s_n)) \geq f\left(\frac{s_1 + \cdots + s_n}{n} \right).$$

This will follow from Jensen's inequality if we can verify that f is a convex function; it is enough to check that $f''(x) > 0$ for all x. In fact,

$$f'(x) = \frac{e^x}{1+e^x} = 1 - \frac{1}{1+e^x}$$

is an increasing function of x, so $f''(x) > 0$ and Jensen's inequality thus yields the desired result. (As long as the a_i are all positive, equality holds when $s_1 = \cdots = s_n$, i.e., when the vectors $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$. Of course other equality cases crop up if some of the a_i vanish, i.e., if $a_1 = b_1 = 0$.)

Fourth solution: We apply induction on n, the case $n = 1$ being evident. First we verify the auxiliary inequality

$$(a^n + b^n)(c^n + d^n)^{n-1} \geq (ac^{n-1} + bd^{n-1})^n$$

for $a, b, c, d \geq 0$. The left side can be written as

$$\begin{aligned} &a^n c^{n(n-1)} + b^n d^{n(n-1)} \\ &+ \sum_{i=1}^{n-1} \binom{n-1}{i} a^n c^{ni} d^{n(n-1-i)} \\ &+ \sum_{i=1}^{n-1} \binom{n-1}{i-1} b^n c^{n(n-i)} d^{n(i-1)}. \end{aligned}$$

$2001 \equiv 1 \pmod 4$, this is impossible. Thus a is odd, and so must divide $1001 = 7 \times 11 \times 13$. Moreover, $a^{n+1} - (a+1)^n \equiv a \pmod 4$, so $a \equiv 1 \pmod 4$.

Of the divisors of $7 \times 11 \times 13$, those congruent to 1 mod 3 are precisely those not divisible by 11 (since 7 and 13 are both congruent to 1 mod 3). Thus a divides 7×13. Now $a \equiv 1 \pmod 4$ is only possible if a divides 13.

We cannot have $a = 1$, since $1 - 2^n \neq 2001$ for any n. Thus the only possibility is $a = 13$. One easily checks that $a = 13, n = 2$ is a solution; all that remains is to check that no other n works. In fact, if $n > 2$, then $13^{n+1} \equiv 2001 \equiv 1 \pmod 8$. But $13^{n+1} \equiv 13 \pmod 8$ since n is even, contradiction. Thus $a = 13, n = 2$ is the unique solution.

Note: once one has that n is even, one can use that $2002 = a^{n+1} + 1 - (a+1)^n$ is divisible by $a+1$ to rule out cases.

A–6 The answer is yes. Consider the arc of the parabola $y = Ax^2$ inside the circle $x^2 + (y-1)^2 = 1$, where we initially assume that $A > 1/2$. This intersects the circle in three points, $(0,0)$ and $(\pm\sqrt{2A-1}/A, (2A-1)/A)$. We claim that for A sufficiently large, the length L of the parabolic arc between $(0,0)$ and $(\sqrt{2A-1}/A, (2A-1)/A)$ is greater than 2, which implies the desired result by symmetry. We express L using the usual formula for arclength:

$$\begin{aligned} L &= \int_0^{\sqrt{2A-1}/A} \sqrt{1 + (2Ax)^2}\,dx \\ &= \frac{1}{2A} \int_0^{2\sqrt{2A-1}} \sqrt{1+x^2}\,dx \\ &= 2 + \frac{1}{2A} \left(\int_0^{2\sqrt{2A-1}} (\sqrt{1+x^2} - x)\,dx - 2 \right), \end{aligned}$$

where we have artificially introduced $-x$ into the integrand in the last step. Now, for $x \geq 0$,

$$\sqrt{1+x^2} - x = \frac{1}{\sqrt{1+x^2}+x} > \frac{1}{2\sqrt{1+x^2}} \geq \frac{1}{2(x+1)};$$

since $\int_0^\infty dx/(2(x+1))$ diverges, so does $\int_0^\infty (\sqrt{1+x^2} - x)\,dx$. Hence, for sufficiently large A, we have $\int_0^{2\sqrt{2A-1}} (\sqrt{1+x^2} - x)\,dx > 2$, and hence $L > 2$.

Note: a numerical computation shows that one must take $A > 34.7$ to obtain $L > 2$, and that the maximum value of L is about 4.0027, achieved for $A \approx 94.1$.

B–1 Let R (resp. B) denote the set of red (resp. black) squares in such a coloring, and for $s \in R \cup B$, let $f(s)n + g(s) + 1$ denote the number written in square s, where $0 \leq f(s), g(s) \leq n-1$. Then it is clear that the value of $f(s)$ depends only on the row of s, while the value of $g(s)$ depends only on the column of s. Since every row contains exactly $n/2$ elements of R and $n/2$ elements of B,

$$\sum_{s \in R} f(s) = \sum_{s \in B} f(s).$$

Similarly, because every column contains exactly $n/2$ elements of R and $n/2$ elements of B,

$$\sum_{s \in R} g(s) = \sum_{s \in B} g(s).$$

It follows that

$$\sum_{s \in R} f(s)n + g(s) + 1 = \sum_{s \in B} f(s)n + g(s) + 1,$$

as desired.

Note: Richard Stanley points out a theorem of Ryser (see Ryser, *Combinatorial Mathematics*, Theorem 3.1) that can also be applied. Namely, if A and B are $0-1$ matrices with the same row and column sums, then there is a sequence of operations on 2×2 matrices of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or vice versa, which transforms A into B. If we identify 0 and 1 with red and black, then the given coloring and the checkerboard coloring both satisfy the sum condition. Since the desired result is clearly true for the checkerboard coloring, and performing the matrix operations does not affect this, the desired result follows in general.

B–2 By adding and subtracting the two given equations, we obtain the equivalent pair of equations

$$\begin{aligned} 2/x &= x^4 + 10x^2y^2 + 5y^4 \\ 1/y &= 5x^4 + 10x^2y^2 + y^4. \end{aligned}$$

Multiplying the former by x and the latter by y, then adding and subtracting the two resulting equations, we obtain another pair of equations equivalent to the given ones,

$$3 = (x+y)^5, \qquad 1 = (x-y)^5.$$

It follows that $x = (3^{1/5}+1)/2$ and $y = (3^{1/5}-1)/2$ is the unique solution satisfying the given equations.

B–3 Since $(k-1/2)^2 = k^2 - k + 1/4$ and $(k+1/2)^2 = k^2 + k + 1/4$, we have that $\langle n \rangle = k$ if and only if

Solutions to the 63rd William Lowell Putnam Mathematical Competition
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Kiran Kedlaya and Lenny Ng

A1 By differentiating $P_n(x)/(x^k-1)^{n+1}$, we find that $P_{n+1}(x) = (x^k-1)P_n'(x) - (n+1)kx^{k-1}P_n(x)$; substituting $x=1$ yields $P_{n+1}(1) = -(n+1)kP_n(1)$. Since $P_0(1) = 1$, an easy induction gives $P_n(1) = (-k)^n n!$ for all $n \geq 0$.

Note: one can also argue by expanding in Taylor series around 1. Namely, we have

$$\frac{1}{x^k-1} = \frac{1}{k(x-1)+\cdots} = \frac{1}{k}(x-1)^{-1} + \cdots,$$

so

$$\frac{d^n}{dx^n}\frac{1}{x^k-1} = \frac{(-1)^n n!}{k(x-1)^{-n-1}}$$

and

$$\begin{aligned} P_n(x) &= (x^k-1)^{n+1}\frac{d^n}{dx^n}\frac{1}{x^k-1} \\ &= (k(x-1)+\cdots)^{n+1} \\ &\quad \left(\frac{(-1)^n n!}{k}(x-1)^{-n-1}+\cdots\right) \\ &= (-k)^n n! + \cdots. \end{aligned}$$

A2 Draw a great circle through two of the points. There are two closed hemispheres with this great circle as boundary, and each of the other three points lies in one of them. By the pigeonhole principle, two of those three points lie in the same hemisphere, and that hemisphere thus contains four of the five given points.

Note: by a similar argument, one can prove that among any $n+3$ points on an n-dimensional sphere, some $n+2$ of them lie on a closed hemisphere. (One cannot get by with only $n+2$ points: put them at the vertices of a regular simplex.) Namely, any n of the points lie on a great sphere, which forms the boundary of two hemispheres; of the remaining three points, some two lie in the same hemisphere.

A3 Note that each of the sets $\{1\}, \{2\}, \dots, \{n\}$ has the desired property. Moreover, for each set S with integer average m that does not contain m, $S \cup \{m\}$ also has average m, while for each set T of more than one element with integer average m that contains m, $T \setminus \{m\}$ also has average m. Thus the subsets other than $\{1\}, \{2\}, \dots, \{n\}$ can be grouped in pairs, so $T_n - n$ is even.

A4 (partly due to David Savitt) Player 0 wins with optimal play. In fact, we prove that Player 1 cannot prevent Player 0 from creating a row of all zeroes, a column of all zeroes, or a 2×2 submatrix of all zeroes. Each of these forces the determinant of the matrix to be zero.

For $i,j = 1,2,3$, let A_{ij} denote the position in row i and column j. Without loss of generality, we may assume that Player 1's first move is at A_{11}. Player 0 then plays at A_{22}:

$$\begin{pmatrix} 1 & * & * \\ * & 0 & * \\ * & * & * \end{pmatrix}$$

After Player 1's second move, at least one of A_{23} and A_{32} remains vacant. Without loss of generality, assume A_{23} remains vacant; Player 0 then plays there.

After Player 1's third move, Player 0 wins by playing at A_{21} if that position is unoccupied. So assume instead that Player 1 has played there. Thus of Player 1's three moves so far, two are at A_{11} and A_{21}. Hence for i equal to one of 1 or 3, and for j equal to one of 2 or 3, the following are both true:

(a) The 2×2 submatrix formed by rows 2 and i and by columns 2 and 3 contains two zeroes and two empty positions.

(b) Column j contains one zero and two empty positions.

Player 0 next plays at A_{ij}. To prevent a zero column, Player 1 must play in column j, upon which Player 0 completes the 2×2 submatrix in (a) for the win.

Note: one can also solve this problem directly by making a tree of possible play sequences. This tree can be considerably collapsed using symmetries: the symmetry between rows and columns, the invariance of the outcome under reordering of rows or columns, and the fact that the scenario after a sequence of moves does not depend on the order of the moves (sometimes called "transposition invariance").

Note (due to Paul Cheng): one can reduce Determinant Tic-Tac-Toe to a variant of ordinary tic-tac-toe. Namely, consider a tic-tac-toe grid labeled as follows:

A_{11}	A_{22}	A_{33}
A_{23}	A_{31}	A_{12}
A_{32}	A_{13}	A_{21}

Solutions to the 67th William Lowell Putnam Mathematical Competition
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Kiran Kedlaya and Lenny Ng

A1 We change to cylindrical coordinates, i.e., we put $r = \sqrt{x^2+y^2}$. Then the given inequality is equivalent to

$$r^2 + z^2 + 8 \leq 6r,$$

or

$$(r-3)^2 + z^2 \leq 1.$$

This defines a solid of revolution (a solid torus); the area being rotated is the disc $(x-3)^2 + z^2 \leq 1$ in the xz-plane. By Pappus's theorem, the volume of this equals the area of this disc, which is π, times the distance through which the center of mass is being rotated, which is $(2\pi)3$. That is, the total volume is $6\pi^2$.

A2 Suppose on the contrary that the set B of values of n for which Bob has a winning strategy is finite; for convenience, we include $n = 0$ in B, and write $B = \{b_1, \dots, b_m\}$. Then for every nonnegative integer n not in B, Alice must have some move on a heap of n stones leading to a position in which the second player wins. That is, every nonnegative integer not in B can be written as $b+p-1$ for some $b \in B$ and some prime p. However, there are numerous ways to show that this cannot happen.

First solution: Let t be any integer bigger than all of the $b \in B$. Then it is easy to write down t consecutive composite integers, e.g., $(t+1)!+2, \dots, (t+1)!+t+1$. Take $n = (t+1)!+t$; then for each $b \in B$, $n-b+1$ is one of the composite integers we just wrote down.

Second solution: Let $p_1, \dots, p_{2m}$ be any prime numbers; then by the Chinese remainder theorem, there exists a positive integer x such that

$$\begin{aligned} x - b_1 &\equiv -1 \pmod{p_1 p_{m+1}} \\ &\dots \\ x - b_n &\equiv -1 \pmod{p_m p_{2m}}. \end{aligned}$$

For each $b \in B$, the unique integer p such that $x = b+p-1$ is divisible by at least two primes, and so cannot itself be prime.

Third solution: (by Catalin Zara) Put $b_1 = 0$, and take $n = (b_2-1)\cdots(b_m-1)$; then n is composite because $3, 8 \in B$, and for any nonzero $b \in B$, $n - b_i + 1$ is divisible by but not equal to $b_i - 1$. (One could also take $n = b_2 \cdots b_m - 1$, so that $n - b_i + 1$ is divisible by b_i.)

A3 We first observe that given any sequence of integers $x_1, x_2, \dots$ satisfying a recursion

$$x_k = f(x_{k-1}, \dots, x_{k-n}) \qquad (k > n),$$

where n is fixed and f is a fixed polynomial of n variables with integer coefficients, for any positive integer N, the sequence modulo N is eventually periodic. This is simply because there are only finitely many possible sequences of n consecutive values modulo N, and once such a sequence is repeated, every subsequent value is repeated as well.

We next observe that if one can rewrite the same recursion as

$$x_{k-n} = g(x_{k-n+1}, \dots, x_k) \qquad (k > n),$$

where g is also a polynomial with integer coefficients, then the sequence extends uniquely to a doubly infinite sequence $\dots, x_{-1}, x_0, x_1, \dots$ which is fully periodic modulo any N. That is the case in the situation at hand, because we can rewrite the given recursion as

$$x_{k-2005} = x_{k+1} - x_k.$$

It thus suffices to find 2005 consecutive terms divisible by N in the doubly infinite sequence, for any fixed N (so in particular for $N = 2006$). Running the recursion backwards, we easily find

$$\begin{gathered} x_1 = x_0 = \cdots = x_{-2004} = 1 \\ x_{-2005} = \cdots = x_{-4009} = 0, \end{gathered}$$

yielding the desired result.

A4 **First solution:** By the linearity of expectation, the average number of local maxima is equal to the sum of the probability of having a local maximum at k over $k = 1, \dots, n$. For $k = 1$, this probability is 1/2: given the pair $\{\pi(1), \pi(2)\}$, it is equally likely that $\pi(1)$ or $\pi(2)$ is bigger. Similarly, for $k = n$, the probability is 1/2. For $1 < k < n$, the probability is 1/3: given the pair $\{\pi(k-1), \pi(k), \pi(k+1)\}$, it is equally likely that any of the three is the largest. Thus the average number of local maxima is

$$2 \cdot \frac{1}{2} + (n-2) \cdot \frac{1}{3} = \frac{n+1}{3}.$$

Second solution: Another way to apply the linearity of expectation is to compute the probability that $i \in \{1, \dots, n\}$ occurs as a local maximum. The most efficient way to do this is to imagine the permutation as consisting of the symbols $1, \dots, n, *$ written in a circle in some order. The number i occurs as a local maximum if the two symbols it is adjacent to both belong to the set $\{*, 1, \dots, i-1\}$. There are $i(i-1)$ pairs of such symbols and $n(n-1)$ pairs in total, so the probability of

i occurring as a local maximum is $i(i-1)/(n(n-1))$, and the average number of local maxima is

$$\begin{aligned}\sum_{i=1}^n \frac{i(i-1)}{n(n-1)} &= \frac{2}{n(n-1)} \sum_{i=1}^n \binom{i}{2} \\ &= \frac{2}{n(n-1)} \binom{n+1}{3} \\ &= \frac{n+1}{3}.\end{aligned}$$

One can obtain a similar (if slightly more intricate) solution inductively, by removing the known local maximum n and splitting into two shorter sequences.

Remark: The usual term for a local maximum in this sense is a *peak*. The complete distribution for the number of peaks is known; Richard Stanley suggests the reference: F. N. David and D. E. Barton, *Combinatorial Chance*, Hafner, New York, 1962, p. 162 and subsequent.

A5 Since the desired expression involves symmetric functions of $a_1, \dots, a_n$, we start by finding a polynomial with $a_1, \dots, a_n$ as roots. Note that

$$1 \pm i \tan\theta = e^{\pm i\theta} \sec\theta$$

so that

$$1 + i\tan\theta = e^{2i\theta}(1 - i\tan\theta).$$

Consequently, if we put $\omega = e^{2in\theta}$, then the polynomial

$$Q_n(x) = (1+ix)^n - \omega(1-ix)^n$$

has among its roots $a_1, \dots, a_n$. Since these are distinct and Q_n has degree n, these must be exactly the roots.

If we write

$$Q_n(x) = c_n x^n + \cdots + c_1 x + c_0,$$

then $a_1 + \cdots + a_n = -c_{n-1}/c_n$ and $a_1 \cdots a_n = -c_0/c_n$, so the ratio we are seeking is c_{n-1}/c_0. By inspection,

$$\begin{aligned} c_{n-1} &= ni^{n-1} - \omega n(-i)^{n-1} = ni^{n-1}(1-\omega) \\ c_0 &= 1 - \omega \end{aligned}$$

so

$$\frac{a_1 + \cdots + a_n}{a_1 \cdots a_n} = \begin{cases} n & n \equiv 1 \pmod 4 \\ -n & n \equiv 3 \pmod 4. \end{cases}$$

Remark: The same argument shows that the ratio between any two *odd* elementary symmetric functions of $a_1, \dots, a_n$ is independent of θ.

A6 **First solution:** (by Daniel Kane) The probability is $1 - \frac{35}{12\pi^2}$. We start with some notation and simplifications. For simplicity, we assume without loss of generality that the circle has radius 1. Let E denote the expected value of a random variable over all choices of P, Q, R. Write $[XYZ]$ for the area of triangle XYZ.

If P, Q, R, S are the four points, we may ignore the case where three of them are collinear, as this occurs with probability zero. Then the only way they can fail to form the vertices of a convex quadrilateral is if one of them lies inside the triangle formed by the other three. There are four such configurations, depending on which point lies inside the triangle, and they are mutually exclusive. Hence the desired probability is 1 minus four times the probability that S lies inside triangle PQR. That latter probability is simply $E([PQR])$ divided by the area of the disc.

Let O denote the center of the circle, and let P', Q', R' be the projections of P, Q, R onto the circle from O. We can write

$$[PQR] = \pm[OPQ] \pm [OQR] \pm [ORP]$$

for a suitable choice of signs, determined as follows. If the points P', Q', R' lie on no semicircle, then all of the signs are positive. If P', Q', R' lie on a semicircle in that order and Q lies inside the triangle OPR, then the sign on $[OPR]$ is positive and the others are negative. If P', Q', R' lie on a semicircle in that order and Q lies outside the triangle OPR, then the sign on $[OPR]$ is negative and the others are positive.

We first calculate

$$E([OPQ] + [OQR] + [ORP]) = 3E([OPQ]).$$

Write $r_1 = OP, r_2 = OQ, \theta = \angle POQ$, so that

$$[OPQ] = \frac{1}{2} r_1 r_2 (\sin\theta).$$

The distribution of r_1 is given by $2r_1$ on $[0,1]$ (e.g., by the change of variable formula to polar coordinates), and similarly for r_2. The distribution of θ is uniform on $[0,\pi]$. These three distributions are independent; hence

$$\begin{aligned} &E([OPQ]) \\ &= \frac{1}{2} \left(\int_0^1 2r^2\,dr \right)^2 \left(\frac{1}{\pi} \int_0^\pi \sin(\theta)\,d\theta \right) \\ &= \frac{4}{9\pi}, \end{aligned}$$

and

$$E([OPQ] + [OQR] + [ORP]) = \frac{4}{3\pi}.$$

We now treat the case where P', Q', R' lie on a semicircle in that order. Put $\theta_1 = \angle POQ$ and $\theta_2 = \angle QOR$; then the distribution of θ_1, θ_2 is uniform on the region

$$0 \le \theta_1, \quad 0 \le \theta_2, \quad \theta_1 + \theta_2 \le \pi.$$

In particular, the distribution on $\theta = \theta_1 + \theta_2$ is $\frac{2\theta}{\pi^2}$ on $[0,\pi]$. Put $r_P = OP, r_Q = OQ, r_R = OR$. Again, the

Now substitute to eliminate evaluations at a/x:

$$f''(x) = -\frac{f'(x)}{x} + \frac{f'(x)^2}{f(x)}.$$

Clear denominators:

$$xf(x)f''(x) + f(x)f'(x) = xf'(x)^2.$$

Divide through by $f(x)^2$ and rearrange:

$$0 = \frac{f'(x)}{f(x)} + \frac{xf''(x)}{f(x)} - \frac{xf'(x)^2}{f(x)^2}.$$

The right side is the derivative of $xf'(x)/f(x)$, so that quantity is constant. That is, for some d,

$$\frac{f'(x)}{f(x)} = \frac{d}{x}.$$

Integrating yields $f(x) = cx^d$, as desired.

B4 **First solution:** Define $f(m,n,k)$ as the number of n-tuples $(x_1, x_2, \dots, x_n)$ of integers such that $|x_1| + \cdots + |x_n| \leq m$ and exactly k of $x_1, \dots, x_n$ are nonzero. To choose such a tuple, we may choose the k nonzero positions, the signs of those k numbers, and then an ordered k-tuple of positive integers with sum $\leq m$. There are $\binom{n}{k}$ options for the first choice, and 2^k for the second. As for the third, we have $\binom{m}{k}$ options by a "stars and bars" argument: depict the k-tuple by drawing a number of stars for each term, separated by bars, and adding stars at the end to get a total of m stars. Then each tuple corresponds to placing k bars, each in a different position behind one of the m fixed stars.

We conclude that

$$f(m,n,k) = 2^k \binom{m}{k}\binom{n}{k} = f(n,m,k);$$

summing over k gives $f(m,n) = f(n,m)$. (One may also extract easily a bijective interpretation of the equality.)

Second solution: (by Greg Kuperberg) It will be convenient to extend the definition of $f(m,n)$ to $m,n \geq 0$, in which case we have $f(0,m) = f(n,0) = 1$.

Let $S_{m,n}$ be the set of n-tuples $(x_1, \dots, x_n)$ of integers such that $|x_1| + \cdots + |x_n| \leq m$. Then elements of $S_{m,n}$ can be classified into three types. Tuples with $|x_1| + \cdots + |x_n| < m$ also belong to $S_{m-1,n}$. Tuples with $|x_1| + \cdots + |x_n| = m$ and $x_n \geq 0$ correspond to elements of $S_{m,n-1}$ by dropping x_n. Tuples with $|x_1| + \cdots + |x_n| = m$ and $x_n < 0$ correspond to elements of $S_{m-1,n-1}$ by dropping x_n. It follows that

$$\begin{aligned} &f(m,n) \\ &= f(m-1,n) + f(m,n-1) + f(m-1,n-1), \end{aligned}$$

so f satisfies a symmetric recurrence with symmetric boundary conditions $f(0,m) = f(n,0) = 1$. Hence f is symmetric.

Third solution: (by Greg Martin) As in the second solution, it is convenient to allow $f(m,0) = f(0,n) = 1$. Define the generating function

$$G(x,y) = \sum_{m=0}^\infty \sum_{n=0}^\infty f(m,n) x^m y^n.$$

As equalities of formal power series (or convergent series on, say, the region $|x|, |y| < \frac{1}{3}$), we have

$$\begin{aligned} G(x,y) &= \sum_{m \geq 0} \sum_{n \geq 0} x^m y^n \sum_{\substack{k_1, \dots, k_n \in \mathbb{Z} \\ |k_1| + \cdots + |k_n| \leq m}} 1 \\ &= \sum_{n \geq 0} y^n \sum_{k_1, \dots, k_n \in \mathbb{Z}} \sum_{m \geq |k_1| + \cdots + |k_n|} x^m \\ &= \sum_{n \geq 0} y^n \sum_{k_1, \dots, k_n \in \mathbb{Z}} \frac{x^{|k_1| + \cdots + |k_n|}}{1-x} \\ &= \frac{1}{1-x} \sum_{n \geq 0} y^n \left(\sum_{k \in \mathbb{Z}} x^{|k|} \right)^n \\ &= \frac{1}{1-x} \sum_{n \geq 0} y^n \left(\frac{1+x}{1-x} \right)^n \\ &= \frac{1}{1-x} \cdot \frac{1}{1 - y(1+x)/(1-x)} \\ &= \frac{1}{1-x-y-xy}. \end{aligned}$$

Since $G(x,y) = G(y,x)$, it follows that $f(m,n) = f(n,m)$ for all $m,n \geq 0$.

B5 **First solution:** Put $Q = x_1^2 + \cdots + x_n^2$. Since Q is homogeneous, P is divisible by Q if and only if each of the homogeneous components of P is divisible by Q. It is thus sufficient to solve the problem in case P itself is homogeneous, say of degree d.

Suppose that we have a factorization $P = Q^m R$ for some $m > 0$, where R is homogeneous of degree d and not divisible by Q; note that the homogeneity implies that

$$\sum_{i=1}^n x_i \frac{\partial R}{\partial x_i} = dR.$$

Write ∇^2 as shorthand for $\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$; then

$$\begin{aligned} 0 &= \nabla^2 P \\ &= 2mnQ^{m-1}R + Q^m \nabla^2 R + 2 \sum_{i=1}^n 2mx_i Q^{m-1} \frac{\partial R}{\partial x_i} \\ &= Q^m \nabla^2 R + (2mn + 4md) Q^{m-1} R. \end{aligned}$$

Since $m > 0$, this forces R to be divisible by Q, contradiction.

Second solution: (by Noam Elkies) Retain notation as in the first solution. Let P_d be the set of homogeneous

If $a-b=0$, then $a=b=\pm\sqrt{2}/2$ and either $f=2+3\sqrt{2}>6.24$, or $f=2-3\sqrt{2}<-2.24$.

If $a=0$, then either $b=-1$ as discussed above, or $b=1$. In the latter case, f blows up as one approaches this point, so there cannot be a global minimum there.

Finally, if $ab-a-b=0$, then

$$a^2b^2=(a+b)^2=2ab+1$$

and so $ab=1\pm\sqrt{2}$. The plus sign is impossible since $|ab|\le 1$, so $ab=1-\sqrt{2}$ and

$$\begin{aligned} f(a,b)&=ab+\frac{1}{ab}+1\\ &=1-2\sqrt{2}>-1.83. \end{aligned}$$

This yields the smallest value of $|f|$ in the list (and indeed no sign crossings are possible), so $2\sqrt{2}-1$ is the desired minimum of $|f|$.

Note: Instead of using the geometry of the graph of f to rule out sign crossings, one can verify explicitly that f cannot take the value 0. In the first solution, note that $c+2/(c-1)=0$ implies $c^2-c+2=0$, which has no real roots. In the second solution, we would have

$$a^2b+ab^2+a+b=-1.$$

Squaring both sides and simplifying yields

$$2a^3b^3+5a^2b^2+4ab=0,$$

whose only real root is $ab=0$. But the cases with $ab=0$ do not yield $f=0$, as verified above.

A4 We split into three cases. Note first that $|A|\ge|a|$, by applying the condition for large x.

Case 1: $B^2-4AC>0$. In this case Ax^2+Bx+C has two distinct real roots r_1 and r_2. The condition implies that ax^2+bx+c also vanishes at r_1 and r_2, so $b^2-4ac>0$. Now

$$\begin{aligned} B^2-4AC&=A^2(r_1-r_2)^2\\ &\ge a^2(r_1-r_2)^2\\ &=b^2-4ac. \end{aligned}$$

Case 2: $B^2-4AC\le 0$ **and** $b^2-4ac\le 0$**.** Assume without loss of generality that $A\ge a>0$, and that $B=0$ (by shifting x). Then $Ax^2+Bx+C\ge ax^2+bx+c\ge 0$ for all x; in particular, $C\ge c\ge 0$. Thus

$$\begin{aligned} 4AC-B^2&=4AC\\ &\ge 4ac\\ &\ge 4ac-b^2. \end{aligned}$$

Alternate derivation (due to Robin Chapman): the ellipse $Ax^2+Bxy+Cy^2=1$ is contained within the ellipse $ax^2+bxy+cy^2=1$, and their respective enclosed areas are $\pi/(4AC-B^2)$ and $\pi/(4ac-b^2)$.

Case 3: $B^2-4AC\le 0$ **and** $b^2-4ac>0$**.** Since Ax^2+Bx+C has a graph not crossing the x-axis, so do $(Ax^2+Bx+C)\pm(ax^2+bx+c)$. Thus

$$\begin{aligned} (B-b)^2-4(A-a)(C-c)&\le 0,\\ (B+b)^2-4(A+a)(C+c)&\le 0 \end{aligned}$$

and adding these together yields

$$2(B^2-4AC)+2(b^2-4ac)\le 0.$$

Hence $b^2-4ac\le 4AC-B^2$, as desired.

A5 **First solution:** We represent a Dyck n-path by a sequence $a_1\cdots a_{2n}$, where each a_i is either $(1,1)$ or $(1,-1)$.

Given an $(n-1)$-path $P=a_1\cdots a_{2n-2}$, we distinguish two cases. If P has no returns of even-length, then let $f(P)$ denote the n-path $(1,1)(1,-1)P$. Otherwise, let $a_ia_{i+1}\cdots a_j$ denote the rightmost even-length return in P, and let $f(P)=(1,1)a_1a_2\cdots a_j(1,-1)a_{j+1}\cdots a_{2n-2}$. Then f clearly maps the set of Dyck $(n-1)$-paths to the set of Dyck n-paths having no even return.

We claim that f is bijective; to see this, we simply construct the inverse mapping. Given an n-path P, let $R=a_ia_{i+1}...a_j$ denote the leftmost return in P, and let $g(P)$ denote the path obtained by removing a_1 and a_j from P. Then evidently $f\circ g$ and $g\circ f$ are identity maps, proving the claim.

Second solution: (by Dan Bernstein) Let C_n be the number of Dyck paths of length n, let O_n be the number of Dyck paths whose final return has odd length, and let X_n be the number of Dyck paths with no return of even length.

We first exhibit a recursion for O_n; note that $O_0=0$. Given a Dyck n-path whose final return has odd length, split it just after its next-to-last return. For some k (possibly zero), this yields a Dyck k-path, an upstep, a Dyck $(n-k-1)$-path whose odd return has even length, and a downstep. Thus for $n\ge 1$,

$$O_n=\sum_{k=0}^{n-1}C_k(C_{n-k-1}-O_{n-k-1}).$$

We next exhibit a similar recursion for X_n; note that $X_0=1$. Given a Dyck n-path with no even return, splitting as above yields for some k a Dyck k-path with no even return, an upstep, a Dyck $(n-k-1)$-path whose final return has even length, then a downstep. Thus for $n\ge 1$,

$$X_n=\sum_{k=0}^{n-1}X_k(C_{n-k-1}-O_{n-k-1}).$$

To conclude, we verify that $X_n = C_{n-1}$ for $n \geq 1$, by induction on n. This is clear for $n = 1$ since $X_1 = C_0 = 1$. Given $X_k = C_{k-1}$ for $k < n$, we have

$$\begin{aligned} X_n &= \sum_{k=0}^{n-1} X_k (C_{n-k-1} - O_{n-k-1}) \\ &= C_{n-1} - O_{n-1} + \sum_{k=1}^{n-1} C_{k-1}(C_{n-k-1} - O_{n-k-1}) \\ &= C_{n-1} - O_{n-1} + O_{n-1} \\ &= C_{n-1}, \end{aligned}$$

as desired.

Note: Since the problem only asked about the *existence* of a one-to-one correspondence, we believe that any proof, bijective or not, that the two sets have the same cardinality is an acceptable solution. (Indeed, it would be highly unusual to insist on using or not using a specific proof technique!) The second solution above can also be phrased in terms of generating functions. Also, the C_n are well-known to equal the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$; the problem at hand is part of a famous exercise in Richard Stanley's *Enumerative Combinatorics, Volume 1* giving 66 combinatorial interpretations of the Catalan numbers.

A6 **First solution:** Yes, such a partition is possible. To achieve it, place each integer into A if it has an even number of 1s in its binary representation, and into B if it has an odd number. (One discovers this by simply attempting to place the first few numbers by hand and noticing the resulting pattern.)

To show that $r_A(n) = r_B(n)$, we exhibit a bijection between the pairs (a_1, a_2) of distinct elements of A with $a_1 + a_2 = n$ and the pairs (b_1, b_2) of distinct elements of B with $b_1 + b_2 = n$. Namely, given a pair (a_1, a_2) with $a_1 + a_2 = n$, write both numbers in binary and find the lowest-order place in which they differ (such a place exists because $a_1 \neq a_2$). Change both numbers in that place and call the resulting numbers b_1, b_2. Then $a_1 + a_2 = b_1 + b_2 = n$, but the parity of the number of 1s in b_1 is opposite that of a_1, and likewise between b_2 and a_2. This yields the desired bijection.

Second solution: (by Micah Smukler) Write $b(n)$ for the number of 1s in the base 2 expansion of n, and $f(n) = (-1)^{b(n)}$. Then the desired partition can be described as $A = f^{-1}(1)$ and $B = f^{-1}(-1)$. Since $f(2n) + f(2n+1) = 0$, we have

$$\sum_{i=0}^{n} f(n) = \begin{cases} 0 & n \text{ odd} \\ f(n) & n \text{ even.} \end{cases}$$

If p, q are both in A, then $f(p) + f(q) = 2$; if p, q are both in B, then $f(p) + f(q) = -2$; if p, q are in different sets, then $f(p) + f(q) = 0$. In other words,

$$2(r_A(n) - r_B(n)) = \sum_{p+q=n, p<q} (f(p) + f(q))$$

and it suffices to show that the sum on the right is always zero. If n is odd, that sum is visibly $\sum_{i=0}^{n} f(i) = 0$. If n is even, the sum equals

$$\left(\sum_{i=0}^{n} f(i) \right) - f(n/2) = f(n) - f(n/2) = 0.$$

This yields the desired result.

Third solution: (by Dan Bernstein) Put $f(x) = \sum_{n \in A} x^n$ and $g(x) = \sum_{n \in B} x^n$; then the value of $r_A(n)$ (resp. $r_B(n)$) is the coefficient of x^n in $f(x)^2 - f(x^2)$ (resp. $g(x)^2 - g(x^2)$). From the evident identities

$$\begin{aligned} \frac{1}{1-x} &= f(x) + g(x) \\ f(x) &= f(x^2) + xg(x^2) \\ g(x) &= g(x^2) + xf(x^2), \end{aligned}$$

we have

$$\begin{aligned} f(x) - g(x) &= f(x^2) - g(x^2) + xg(x^2) - xf(x^2) \\ &= (1-x)(f(x^2) - g(x^2)) \\ &= \frac{f(x^2) - g(x^2)}{f(x) + g(x)}. \end{aligned}$$

We deduce that $f(x)^2 - g(x)^2 = f(x^2) - g(x^2)$, yielding the desired equality.

Note: This partition is actually unique, up to interchanging A and B. More precisely, the condition that $0 \in A$ and $r_A(n) = r_B(n)$ for $n = 1, \dots, m$ uniquely determines the positions of $0, \dots, m$. We see this by induction on m: given the result for $m - 1$, switching the location of m changes $r_A(m)$ by one and does not change $r_B(m)$, so it is not possible for both positions to work. Robin Chapman points out this problem is solved in D.J. Newman's *Analytic Number Theory* (Springer, 1998); in that solution, one uses generating functions to find the partition and establish its uniqueness, not just verify it.

B1 No, there do not.

First solution: Suppose the contrary. By setting $y = -1, 0, 1$ in succession, we see that the polynomials $1 - x + x^2, 1, 1 + x + x^2$ are linear combinations of $a(x)$ and $b(x)$. But these three polynomials are linearly independent, so cannot all be written as linear combinations of two other polynomials, contradiction.

Alternate formulation: the given equation expresses a diagonal matrix with $1, 1, 1$ and zeroes on the diagonal, which has rank 3, as the sum of two matrices of rank 1. But the rank of a sum of matrices is at most the sum of the ranks of the individual matrices.

A5 **First solution:** First recall that any graph with n vertices and e edges has at least $n-e$ connected components (add each edge one at a time, and note that it reduces the number of components by at most 1). Now imagine the squares of the checkerboard as a graph, whose vertices are connected if the corresponding squares share a side and are the same color. Let A be the number of edges in the graph, and let B be the number of 4-cycles (formed by monochromatic 2×2 squares). If we remove the bottom edge of each 4-cycle, the resulting graph has the same number of connected components as the original one; hence this number is at least

$$mn - A + B.$$

By the linearity of expectation, the expected number of connected components is at least

$$mn - E(A) + E(B).$$

Moreover, we may compute $E(A)$ by summing over the individual pairs of adjacent squares, and we may compute $E(B)$ by summing over the individual 2×2 squares. Thus

$$\begin{aligned} E(A) &= \frac{1}{2}(m(n-1)+(m-1)n), \\ E(B) &= \frac{1}{8}(m-1)(n-1), \end{aligned}$$

and so the expected number of components is at least

$$\begin{aligned} &mn - \frac{1}{2}(m(n-1)+(m-1)n) + \frac{1}{8}(m-1)(n-1) \\ &= \frac{mn+3m+3n+1}{8} > \frac{mn}{8}. \end{aligned}$$

Remark: A "dual" approach is to consider the graph whose vertices are the corners of the squares of the checkerboard, with two vertices joined if they are adjacent and the edge between then does not separate two squares of the same color. In this approach, the 4-cycles become isolated vertices, and the bound on components is replaced by a call to Euler's formula relating the vertices, edges and faces of a planar figure. (One must be careful, however, to correctly handle faces which are not simply connected.)

Second solution: (by Noam Elkies) Number the squares of the checkerboard $1,\dots,mn$ by numbering the first row from left to right, then the second row, and so on. We prove by induction on i that if we just consider the figure formed by the first i squares, its expected number of monochromatic components is at least $i/8$. For $i=1$, this is clear.

Suppose the i-th square does not abut the left edge or the top row of the board. Then we may divide into three cases.

- With probability $1/4$, the i-th square is opposite in color from the adjacent squares directly above and to the left of it. In this case adding the i-th square adds one component.
- With probability $1/8$, the i-th square is the same in color as the adjacent squares directly above and to the left of it, but opposite in color from its diagonal neighbor above and to the left. In this case, adding the i-th square either removes a component or leaves the number unchanged.
- In all other cases, the number of components remains unchanged upon adding the i-th square.

Hence adding the i-th square increases the expected number of components by $1/4 - 1/8 = 1/8$.

If the i-th square does abut the left edge of the board, the situation is even simpler: if the i-th square differs in color from the square above it, one component is added, otherwise the number does not change. Hence adding the i-th square increases the expected number of components by $1/2$; likewise if the i-th square abuts the top edge of the board. Thus the expected number of components is at least $i/8$ by induction, as desired.

Remark: Some solvers attempted to consider adding one row at a time, rather than one square; this must be handled with great care, as it is possible that the number of components can drop rather precipitously upon adding an entire row.

A6 By approximating each integral with a Riemann sum, we may reduce to proving the discrete analogue: for $x_{ij}\in\mathbb{R}$ for $i,j=1,\dots,n$,

$$\begin{aligned} &n\sum_{i=1}^n\left(\sum_{j=1}^n x_{ij}\right)^2 + n\sum_{j=1}^n\left(\sum_{i=1}^n x_{ij}\right)^2 \\ &\leq \left(\sum_{i=1}^n\sum_{j=1}^n x_{ij}\right)^2 + n^2\sum_{i=1}^n\sum_{j=1}^n x_{ij}^2. \end{aligned}$$

The difference between the right side and the left side is

$$\frac{1}{4}\sum_{i,j,k,l=1}^n (x_{ij}+x_{kl}-x_{il}-x_{kj})^2,$$

which is evidently nonnegative. If you prefer not to discretize, you may rewrite the original inequality as

$$\int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,z,w)^2\,dx\,dy\,dz\,dw \geq 0$$

for

$$F(x,y,z,w) = f(x,y)+f(z,w)-f(x,w)-f(z,y).$$