Solutions to the 60th William Lowell Putnam Mathematical Competition
Saturday, December 4, 1999

Manjul Bhargava, Kiran Kedlaya, and Lenny Ng

A–1 Note that if $r(x)$ and $s(x)$ are any two functions, then

$$\max(r, s) = (r + s + |r - s|)/2.$$

Therefore, if $F(x)$ is the given function, we have

$$\begin{aligned} F(x) &= \max\{-3x-3, 0\} - \max\{5x, 0\} + 3x + 2 \\ &= (-3x - 3 + |3x+3|)/2 \\ &\quad - (5x + |5x|)/2 + 3x + 2 \\ &= |(3x+3)/2| - |5x/2| - x + \frac{1}{2}, \end{aligned}$$

so we may set $f(x) = (3x+3)/2$, $g(x) = 5x/2$, and $h(x) = -x + \frac{1}{2}$.

A–2 First solution: First factor $p(x) = q(x)r(x)$, where q has all real roots and r has all complex roots. Notice that each root of q has even multiplicity, otherwise p would have a sign change at that root. Thus $q(x)$ has a square root $s(x)$.

Now write $r(x) = \prod_{j=1}^k (x - a_j)(x - \overline{a_j})$ (possible because r has roots in complex conjugate pairs). Write $\prod_{j=1}^k (x - a_j) = t(x) + iu(x)$ with t, x having real coefficients. Then for x real,

$$\begin{aligned} p(x) &= q(x)r(x) \\ &= s(x)^2 (t(x) + iu(x))(\overline{t(x) + iu(x)}) \\ &= (s(x)t(x))^2 + (s(x)u(x))^2. \end{aligned}$$

(Alternatively, one can factor $r(x)$ as a product of quadratic polynomials with real coefficients, write each as a sum of squares, then multiply together to get a sum of many squares.)

Second solution: We proceed by induction on the degree of p, with base case where p has degree 0. As in the first solution, we may reduce to a smaller degree in case p has any real roots, so assume it has none. Then $p(x) > 0$ for all real x, and since $p(x) \to \infty$ for $x \to \pm\infty$, p has a minimum value c. Now $p(x) - c$ has real roots, so as above, we deduce that $p(x) - c$ is a sum of squares. Now add one more square, namely $(\sqrt{c})^2$, to get $p(x)$ as a sum of squares.

A–3 First solution: Computing the coefficient of x^{n+1} in the identity $(1 - 2x - x^2)\sum_{m=0}^\infty a_m x^m = 1$ yields the recurrence $a_{n+1} = 2a_n + a_{n-1}$; the sequence $\{a_n\}$ is then characterized by this recurrence and the initial conditions $a_0 = 1, a_1 = 2$.

Define the sequence $\{b_n\}$ by $b_{2n} = a_{n-1}^2 + a_n^2,\ b_{2n+1} = a_n(a_{n-1} + a_{n+1})$. Then

$$\begin{aligned} 2b_{2n+1} + b_{2n} &= 2a_n a_{n+1} + 2a_{n-1}a_n + a_{n-1}^2 + a_n^2 \\ &= 2a_n a_{n+1} + a_{n-1}a_{n+1} + a_n^2 \\ &= a_{n+1}^2 + a_n^2 = b_{2n+2}, \end{aligned}$$

and similarly $2b_{2n} + b_{2n-1} = b_{2n+1}$, so that $\{b_n\}$ satisfies the same recurrence as $\{a_n\}$. Since further $b_0 = 1, b_1 = 2$ (where we use the recurrence for $\{a_n\}$ to calculate $a_{-1} = 0$), we deduce that $b_n = a_n$ for all n. In particular, $a_n^2 + a_{n+1}^2 = b_{2n+2} = a_{2n+2}$.

Second solution: Note that

$$\begin{aligned} &\frac{1}{1 - 2x - x^2} \\ &\quad = \frac{1}{2\sqrt{2}} \left(\frac{\sqrt{2}+1}{1 - (1+\sqrt{2})x} + \frac{\sqrt{2}-1}{1 - (1-\sqrt{2})x} \right) \end{aligned}$$

and that

$$\frac{1}{1 + (1 \pm \sqrt{2})x} = \sum_{n=0}^\infty (1 \pm \sqrt{2})^n x^n,$$

so that

$$a_n = \frac{1}{2\sqrt{2}} \left((\sqrt{2}+1)^{n+1} - (1-\sqrt{2})^{n+1} \right).$$

A simple computation (omitted here) now shows that $a_n^2 + a_{n+1}^2 = a_{2n+2}$.

Third solution (by Richard Stanley): Let A be the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. A simple induction argument shows that

$$A^{n+2} = \begin{pmatrix} a_n & a_{n+1} \\ a_{n+1} & a_{n+2} \end{pmatrix}.$$

The desired result now follows from comparing the top left corner entries of the equality $A^{n+2}A^{n+2} = A^{2n+4}$.

A–4 Denote the series by S, and let $a_n = 3^n/n$. Note that

$$\begin{aligned} S &= \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{1}{a_m(a_m + a_n)} \\ &= \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{1}{a_n(a_m + a_n)}, \end{aligned}$$

543. Find all functions $f : (0, \infty) \to (0, \infty)$ subject to the conditions
(i) $f(f(f(x))) + 2x = f(3x)$, for all $x > 0$;
(ii) $\lim_{x\to\infty}(f(x) - x) = 0$.

544. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation

$$g(x - y) = g(x)g(y) + f(x)f(y)$$

for x and y in $\mathbb{R}$, and that $f(t) = 1$ and $g(t) = 0$ for some $t \neq 0$. Prove that f and g satisfy

$$g(x + y) = g(x)g(y) - f(x)f(y)$$

and

$$f(x \pm y) = f(x)g(y) \pm g(x)f(y)$$

for all real x and y.

A famous functional equation, which carries the name of Cauchy, is

$$f(x + y) = f(x) + f(y).$$

We are looking for solutions $f : \mathbb{R} \to \mathbb{R}$.

It is straightforward that $f(2x) = 2f(x)$, and inductively $f(nx) = nf(x)$. Setting $y = nx$, we obtain $f(\frac{1}{n}y) = \frac{1}{n}f(y)$. In general, if m, n are positive integers, then $f(\frac{m}{n}) = mf(\frac{1}{n}) = \frac{m}{n}f(1)$.

On the other hand, $f(0) = f(0) + f(0)$ implies $f(0) = 0$, and $0 = f(0) = f(x) + f(-x)$ implies $f(-x) = -f(x)$. We conclude that for any rational number x, $f(x) = f(1)x$.

If f is continuous, then the linear functions of the form

$$f(x) = cx,$$

where $c \in \mathbb{R}$, are the only solutions. That is because a solution is linear when restricted to rational numbers and therefore must be linear on the whole real axis. Even if we assume the solution f to be continuous at just one point, it still is linear. Indeed, because $f(x + y)$ is the translate of $f(x)$ by $f(y)$, f must be continuous everywhere.

But if we do not assume continuity, the situation is more complicated. In set theory there is an independent statement called the *axiom of choice*, which postulates that given a family of nonempty sets $(A_i)_{i\in I}$, there is a function $f : I \to \cup_i A_i$ with $f(i) \in A_i$. In other words, it is possible to select one element from each set.

Real numbers form an infinite-dimensional vector space over the rational numbers (vectors are real numbers, scalars are rational numbers). A corollary of the axiom of choice (Zorn's lemma) implies the existence of a basis for this vector space. If $(e_i)_{i\in I}$ is this basis, then any real number x can be expressed uniquely as

$$x = r_1 e_{i_1} + r_2 e_{i_2} + \cdots + r_n e_{i_n},$$

where $r_1, r_2, \ldots, r_n$ are nonzero rational numbers. To obtain a solution to Cauchy's equation, make any choice for $f(e_i)$, $i \in I$, and then extend f to all reals in such a way that it is linear over the rationals. Most of these functions are discontinuous. As an example, for a basis that contains the real number 1, set $f(1) = 1$ and $f(e_i) = 0$ for all other basis elements. Then this function is not continuous.

The problems below are all about Cauchy's equation for continuous functions.

545. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous nonzero function, satisfying the equation

$$f(x+y) = f(x)f(y), \quad \text{for all } x, y \in \mathbb{R}.$$

Prove that there exists $c > 0$ such that $f(x) = c^x$ for all $x \in \mathbb{R}$.

546. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) = f(x) + f(y) + f(x)f(y), \quad \text{for all } x, y \in \mathbb{R}.$$

547. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) = \frac{f(x)+f(y)}{1+f(x)f(y)}, \quad \text{for all } x, y \in \mathbb{R}.$$

548. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$f(xy) = xf(y) + yf(x), \quad \text{for all } x, y \in \mathbb{R}.$$

549. Find the continuous functions $\phi, f, g, h : \mathbb{R} \to \mathbb{R}$ satisfying

$$\phi(x+y+z) = f(x) + g(y) + h(z),$$

for all real numbers x, y, z.

550. Given a positive integer $n \geq 2$, find the continuous functions $f : \mathbb{R} \to \mathbb{R}$, with the property that for any real numbers $x_1, x_2, \ldots, x_n$,

$$\sum_i f(x_i) - \sum_{i<j} f(x_i + x_j) + \sum_{i<j<k} f(x_i + x_j + x_k) + \cdots + (-1)^{n-1} f(x_1 + x_2 + \cdots + x_n) = 0.$$

544. We should keep in mind that $f(x) = \sin x$ and $g(x) = \cos x$ satisfy the condition. As we proceed with the solution to the problem, we try to recover some properties of $\sin x$ and $\cos x$. First, note that the condition $f(t) = 1$ and $g(t) = 0$ for some $t \neq 0$ implies $g(0) = 1$; hence g is nonconstant. Also, $0 = g(t) = g(0)g(t) + f(0)f(t) = f(0)$; hence f is nonconstant. Substituting $x = 0$ in the relation yields $g(-y) = g(y)$, so g is even.

Substituting $y = t$, we obtain $g(x - t) = f(x)$, with its shifted version $f(x + t) = g(x)$. Since g is even, it follows that $f(-x) = g(x + t)$. Now let us combine these facts to obtain

$$\begin{aligned} f(x-y) &= g(x-y-t) = g(x)g(y+t) + f(x)f(y+t) \\ &= g(x)f(-y) + f(x)g(y). \end{aligned}$$

Change y to $-y$ to obtain $f(x + y) = f(x)g(y) + g(x)f(y)$ (the addition formula for sine).

The remaining two identities are consequences of this and the fact that f is odd. Let us prove this fact. From $g(x - (-y)) = g(x + y) = g(-x - y)$, we obtain

$$f(x)f(-y) = f(y)f(-x)$$

for all x and y in $\mathbb{R}$. Setting $y = t$ and $x = -t$ yields $f(-t)^2 = 1$, so $f(-t) = \pm 1$. The choice $f(-t) = 1$ gives $f(x) = f(x)f(-t) = f(-x)f(t) = f(-x)$; hence f is even. But then

$$f(x-y) = f(x)g(-y) + g(x)f(-y) = f(x)g(y) + g(x)f(y) = f(x+y),$$

for all x and y. For $x = \frac{z+w}{2}$, $y = \frac{z-w}{2}$, we have $f(z) = f(w)$, and so f is constant, a contradiction. For $f(-t) = -1$, we obtain $f(-x) = -f(-x)f(-t) = -f(x)f(t) = -f(x)$; hence f is odd. It is now straightforward that

$$f(x-y) = f(x)g(y) + g(x)f(-y) = f(x)g(y) - g(x)f(y)$$

and

$$g(x+y) = g(x-(-y)) = g(x)g(-y) + f(x)f(-y) = g(x)g(y) - f(x)f(y),$$

where in the last equality we also used the fact, proved above, that g is even.

(*American Mathematical Monthly*, proposed by V.L. Klee, solution by P.L. Kannappan)

545. Because $f(x) = f^2(x/2) > 0$, the function $g(x) = \ln f(x)$ is well defined. It satisfies Cauchy's equation and is continuous; therefore, $g(x) = \alpha x$ for some constant α. We obtain $f(x) = c^x$, with $c = e^\alpha$.

We conclude our discussion about functional equations with another instance in which continuity is important. The intermediate value property implies that a one-to-one continuous function is automatically monotonic. So if we can read from a functional equation that a function, which is assumed to be continuous, is also one-to-one, then we know that the function is monotonic, a much more powerful property to be used in the solution.

Example. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $(f \circ f \circ f)(x) = x$ for all $x \in \mathbb{R}$.

Solution. For any $x \in \mathbb{R}$, the image of $f(f(x))$ through f is x. This shows that f is onto. Also, if $f(x_1) = f(x_2)$ then $x_1 = f(f(f(x_1))) = f(f(f(x_2))) = x_2$, which shows that f is one-to-one. Therefore, f is a continuous bijection, so it must be strictly monotonic. If f is decreasing, then $f \circ f$ is increasing and $f \circ f \circ f$ is decreasing, contradicting the hypothesis. Therefore, f is strictly increasing.

Fix x and let us compare $f(x)$ and x. There are three possibilities. First, we could have $f(x) > x$. Monotonicity implies $f(f(x)) > f(x) > x$, and applying f again, we have $x = f(f(f(x))) > f(f(x)) > f(x) > x$, impossible. Or we could have $f(x) < x$, which then implies $f(f(x)) < f(x) < x$, and $x = f(f(f(x))) < f(f(x)) < f(x) < x$, which again is impossible. Therefore, $f(x) = x$. Since x was arbitrary, this shows that the unique solution to the functional equation is the identity function $f(x) = x$. □

551. Do there exist continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^2$ and $g(f(x)) = x^3$ for all $x \in \mathbb{R}$?

552. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f(f(x)) - 2f(x) + x = 0, \quad \text{for all } x \in \mathbb{R}.$$

3.4.2 Ordinary Differential Equations of the First Order

Of far greater importance than functional equations are the differential equations, because practically every evolutionary phenomenon of the real world can be modeled by a differential equation. This section is about first-order ordinary differential equations, namely equations expressed in terms of an unknown one-variable function, its derivative, and the variable. In their most general form, they are written as $F(x, y, y') = 0$, but we will be concerned with only two classes of such equations: separable and exact.

An equation is called separable if it is of the form $\frac{dy}{dx} = f(x)g(y)$. In this case we formally separate the variables and write

$$\int \frac{dy}{g(y)} = \int f(x)dx.$$

After integration, we obtain the solution in implicit form, as an algebraic relation between x and y. Here is a problem of I.V. Maftei from the 1971 Romanian Mathematical Olympiad that applies this method.

$$= \oint_{C_2} \frac{\partial}{\partial z'}((x'-x)^2 + (y'-y)^2 + (z'-z)^2)^{-3/2} dz' = 0,$$

where the last equality is a consequence of the fundamental theorem of calculus. Of the two, only $\frac{\partial Q}{\partial x}$ has a dx' in it, and that part is

$$3\oint_{C_2}((x-x')^2 + (y-y')^2 + (z-z')^2)^{-5/2}(x-x')(z-z')dx'$$
$$= \oint_{C_2} \frac{\partial}{\partial x'} \frac{z-z'}{((x-x')^2 + (y-y')^2 + (z-z')^2)^{3/2}} dx' = 0.$$

The term involving dy' is treated similarly. The conclusion follows.

Remark. The linking number is, in fact, an integer, which measures the number of times the curves wind around each other. It was defined by C.F. Gauss, who used it to decide, based on astronomical observations, whether the orbits of certain asteroids were winding around the orbit of the earth.

535. Plugging in $x = y$, we find that $f(0) = 0$, and plugging in $x = -1$, $y = 0$, we find that $f(1) = -f(-1)$. Also, plugging in $x = a$, $y = 1$, and then $x = a$, $y = -1$, we obtain

$$f(a^2 - 1) = (a-1)(f(a) + f(1)),$$
$$f(a^2 - 1) = (a+1)(f(a) - f(1)).$$

Equating the right-hand sides and solving for $f(a)$ gives $f(a) = f(1)a$ for all a.

So any such function is linear. Conversely, a function of the form $f(x) = kx$ clearly satisfies the equation.

(Korean Mathematical Olympiad, 2000)

536. Replace z by $1 - z$ to obtain

$$f(1-z) + (1-z)f(z) = 2 - z.$$

Combine this with $f(z) + zf(1-z) = 1 + z$, and eliminate $f(1-z)$ to obtain

$$(1 - z + z^2)f(z) = 1 - z + z^2.$$

Hence $f(z) = 1$ for all z except maybe for $z = e^{\pm\pi i/3}$, when $1 - z + z^2 = 0$. For $\alpha = e^{i\pi/3}$, $\bar{\alpha} = \alpha^2 = 1 - \alpha$; hence $f(\alpha) + \alpha f(\bar{\alpha}) = 1 + \alpha$. We therefore have only one constraint, namely $f(\bar{\alpha}) = [1 + \alpha - f(\alpha)]/\alpha = \bar{\alpha} + 1 - \bar{\alpha}f(\alpha)$. Hence the solution to the functional equation is of the form

$$f(z) = 1 \quad \text{for } z \neq e^{\pm i\pi/3},$$

a, b are complex numbers satisfying $a^2 + b^2 = 1$. Therefore $b = (1 - a^2)^{1/2}$ (where the exponent 1/2 means one of the two complex numbers whose square is $1 - a^2$). We conclude that the matrices satisfying $A = A' = A^{-1}$ are $\pm I$ and $\begin{pmatrix} a & (1-a^2)^{1/2} \\ (1-a^2)^{1/2} & -a \end{pmatrix}$ where a is any complex number.

4. Set $R = e^{2\pi i/7} = \cos 2\pi/7 + i \sin 2\pi/7$. Since $R \neq 1$ and $R^7 = 1$, we see that $1 + R + \cdots + R^6 = 0$. Now for n an integer, $R^n = \cos 2n\pi/7 + i \sin 2n\pi/7$. Thus by taking the real parts and using $\cos(2\pi - x) = \cos x$, $\cos(\pi - x) = -\cos x$, we obtain

$$1 + 2\cos\frac{2\pi}{7} - 2\cos\frac{\pi}{7} - 2\cos\frac{3\pi}{7} = 0.$$

Since $\cos\pi/7 + \cos 3\pi/7 = 2\cos(2\pi/7)(\cos\pi/7)$, the above becomes

$$4\cos\frac{2\pi}{7}\cos\frac{\pi}{7} - 2\cos\frac{2\pi}{7} = -1.$$

Finally $\cos(2\pi/7) = 2\cos^2(\pi/7) - 1$, hence $(2\cos^2(\pi/7) - 1)(4\cos(\pi/7) - 2) = -1$ and we conclude that $8\cos^3(\pi/7) - 4\cos^2(\pi/7) - 4\cos(\pi/7) = -1$. Therefore the rational number required is $-1/4$.

5. Since $\angle ABC + \angle PQC = 90$ and $\angle ACB + \angle PRB = 90$, we see that $\angle QPR = \angle ABC + \angle ACB$. Now X, Y, Z being the midpoints of BC, CA, AB respectively tells us that AY is parallel to ZX, AZ is parallel to XY, and BX is parallel to YZ. We deduce that $\angle ZXY = \angle BAC$ and hence $\angle QPR + \angle ZXY = 180$. Therefore the points P, Z, X, Y lie on a circle and we deduce that $\angle QPX = \angle ZYX$. Using BZ parallel to XY and BX parallel to ZY from above, we conclude that $\angle ZYX = \angle ABC$. Therefore $\angle QPX + \angle PQX = \angle ABC + \angle PQX = 90$ and the result follows.

6. Set $g = f^2$. Note that g is continuous, $g^3(x) = x$ for all x, and $f(x) = x$ for all x if and only if $g(x) = x$ for all x. Suppose $y \in [0, 1]$ and $f(y) \neq y$. Then the numbers $y, f(y), f^2(y)$ are distinct. Replacing y with $f(y)$ or $f^2(y)$ and f with g if necessary, we may assume that $y < f(y) < f^2(y)$. Choose $a \in (f(y), f^2(y))$. Since f is continuous, there exists $p \in (y, f(y))$ and $q \in (f(y), f^2(y))$ such that $f(p) = a = f(q)$. Thus $f(p) = f(q)$, hence $f^3(p) = f^3(q)$ and we deduce that $p = q$. This is a contradiction because $p < f(y) < q$, and the result follows.

at Q_2 and proceeding counterclockwise must contain all of $Q_3, \dots, Q_n$, while the open semicircle starting at Q_i and proceeding counterclockwise must contain $Q_{i+1}, \dots, Q_n, Q_1, \dots, Q_{i-1}$. Thus two open semicircles cover the entire circle, contradiction.

It follows that if the polygon has at least one acute angle, then it has either one acute angle or two acute angles occurring consecutively. In particular, there is a unique pair of consecutive vertices Q_1, Q_2 in counterclockwise order for which $\angle Q_2$ is acute and $\angle Q_1$ is not acute. Then the remaining points all lie in the arc from the antipode of Q_1 to Q_1, but Q_2 cannot lie in the arc, and the remaining points cannot all lie in the arc from the antipode of Q_1 to the antipode of Q_2. Given the choice of Q_1, Q_2, let x be the measure of the counterclockwise arc from Q_1 to Q_2; then the probability that the other points fall into position is $2^{-n+2} - x^{n-2}$ if $x \leq 1/2$ and 0 otherwise.

Hence the probability that the polygon has at least one acute angle with a *given* choice of which two points will act as Q_1 and Q_2 is

$$\int_0^{1/2} (2^{-n+2} - x^{n-2})\,dx = \frac{n-2}{n-1} 2^{-n+1}.$$

Since there are $n(n-1)$ choices for which two points act as Q_1 and Q_2, the probability of at least one acute angle is $n(n-2)2^{-n+1}$.

Second solution: (by Calvin Lin) As in the first solution, we may compute the probability that for a particular one of the points Q_1, the angle at Q_1 is not acute but the following angle is, and then multiply by n. Imagine picking the points by first choosing Q_1, then picking $n-1$ *pairs* of antipodal points and then picking one member of each pair. Let $R_2, \dots, R_n$ be the points of the pairs which lie in the semicircle, taken in order away from Q_1, and let $S_2, \dots, S_n$ be the antipodes of these. Then to get the desired situation, we must choose from the pairs to end up with all but one of the S_i, and we cannot take R_n and the other S_i or else $\angle Q_1$ will be acute. That gives us $(n-2)$ good choices out of 2^{n-1}; since we could have chosen Q_1 to be any of the n points, the probability is again $n(n-2)2^{-n+1}$.

B1 Take $P(x,y) = (y-2x)(y-2x-1)$. To see that this works, first note that if $m = \lfloor a \rfloor$, then $2m$ is an integer less than or equal to $2a$, so $2m \leq \lfloor 2a \rfloor$. On the other hand, $m+1$ is an integer strictly greater than a, so $2m+2$ is an integer strictly greater than $2a$, so $\lfloor 2a \rfloor \leq 2m+1$.

B2 By the arithmetic-harmonic mean inequality or the Cauchy-Schwarz inequality,

$$(k_1 + \cdots + k_n)\left(\frac{1}{k_1} + \cdots + \frac{1}{k_n}\right) \geq n^2.$$

We must thus have $5n - 4 \geq n^2$, so $n \leq 4$. Without loss of generality, we may suppose that $k_1 \leq \cdots \leq k_n$.

If $n = 1$, we must have $k_1 = 1$, which works. Note that hereafter we cannot have $k_1 = 1$.

If $n = 2$, we have $(k_1, k_2) \in \{(2,4),(3,3)\}$, neither of which work.

If $n = 3$, we have $k_1 + k_2 + k_3 = 11$, so $2 \leq k_1 \leq 3$. Hence $(k_1,k_2,k_3) \in \{(2,2,7),(2,3,6),(2,4,5),(3,3,5),(3,4,4)\}$, and only $(2,3,6)$ works.

If $n = 4$, we must have equality in the AM-HM inequality, which only happens when $k_1 = k_2 = k_3 = k_4 = 4$.

Hence the solutions are $n = 1$ and $k_1 = 1$, $n = 3$ and (k_1,k_2,k_3) is a permutation of $(2,3,6)$, and $n = 4$ and $(k_1,k_2,k_3,k_4) = (4,4,4,4)$.

Remark: In the cases $n = 2, 3$, Greg Kuperberg suggests the alternate approach of enumerating the solutions of $1/k_1 + \cdots + 1/k_n = 1$ with $k_1 \leq \cdots \leq k_n$. This is easily done by proceeding in lexicographic order: one obtains $(2,2)$ for $n = 2$, and $(2,3,6),(2,4,4),(3,3,3)$ for $n = 3$, and only $(2,3,6)$ contributes to the final answer.

B3 **First solution:** The functions are precisely $f(x) = cx^d$ for $c, d > 0$ arbitrary except that we must take $c = 1$ in case $d = 1$. To see that these work, note that $f'(a/x) = dc(a/x)^{d-1}$ and $x/f(x) = 1/(cx^{d-1})$, so the given equation holds if and only if $dc^2a^{d-1} = 1$. If $d \neq 1$, we may solve for a no matter what c is; if $d = 1$, we must have $c = 1$. (Thanks to Brad Rodgers for pointing out the $d = 1$ restriction.)

To check that these are all solutions, put $b = \log(a)$ and $y = \log(a/x)$; rewrite the given equation as

$$f(e^{b-y})f'(e^y) = e^{b-y}.$$

Put

$$g(y) = \log f(e^y);$$

then the given equation rewrites as

$$g(b-y) + \log g'(y) + g(y) - y = b - y,$$

or

$$\log g'(y) = b - g(y) - g(b-y).$$

By the symmetry of the right side, we have $g'(b-y) = g'(y)$. Hence the function $g(y) + g(b-y)$ has zero derivative and so is constant, as then is $g'(y)$. From this we deduce that $f(x) = cx^d$ for some c, d, both necessarily positive since $f'(x) > 0$ for all x.

Second solution: (suggested by several people) Substitute a/x for x in the given equation:

$$f'(x) = \frac{a}{xf(a/x)}.$$

Differentiate:

$$f''(x) = -\frac{a}{x^2 f(a/x)} + \frac{a^2 f'(a/x)}{x^3 f(a/x)^2}.$$

Now substitute to eliminate evaluations at a/x:

$$f''(x) = -\frac{f'(x)}{x} + \frac{f'(x)^2}{f(x)}.$$

Clear denominators:

$$xf(x)f''(x) + f(x)f'(x) = xf'(x)^2.$$

Divide through by $f(x)^2$ and rearrange:

$$0 = \frac{f'(x)}{f(x)} + \frac{xf''(x)}{f(x)} - \frac{xf'(x)^2}{f(x)^2}.$$

The right side is the derivative of $xf'(x)/f(x)$, so that quantity is constant. That is, for some d,

$$\frac{f'(x)}{f(x)} = \frac{d}{x}.$$

Integrating yields $f(x) = cx^d$, as desired.

B4 **First solution:** Define $f(m,n,k)$ as the number of n-tuples $(x_1, x_2, \dots, x_n)$ of integers such that $|x_1| + \cdots + |x_n| \leq m$ and exactly k of $x_1, \dots, x_n$ are nonzero. To choose such a tuple, we may choose the k nonzero positions, the signs of those k numbers, and then an ordered k-tuple of positive integers with sum $\leq m$. There are $\binom{n}{k}$ options for the first choice, and 2^k for the second. As for the third, we have $\binom{m}{k}$ options by a "stars and bars" argument: depict the k-tuple by drawing a number of stars for each term, separated by bars, and adding stars at the end to get a total of m stars. Then each tuple corresponds to placing k bars, each in a different position behind one of the m fixed stars.

We conclude that

$$f(m,n,k) = 2^k \binom{m}{k} \binom{n}{k} = f(n,m,k);$$

summing over k gives $f(m,n) = f(n,m)$. (One may also extract easily a bijective interpretation of the equality.)

Second solution: (by Greg Kuperberg) It will be convenient to extend the definition of $f(m,n)$ to $m,n \geq 0$, in which case we have $f(0,m) = f(n,0) = 1$.

Let $S_{m,n}$ be the set of n-tuples $(x_1, \dots, x_n)$ of integers such that $|x_1| + \cdots + |x_n| \leq m$. Then elements of $S_{m,n}$ can be classified into three types. Tuples with $|x_1| + \cdots + |x_n| < m$ also belong to $S_{m-1,n}$. Tuples with $|x_1| + \cdots + |x_n| = m$ and $x_n \geq 0$ correspond to elements of $S_{m,n-1}$ by dropping x_n. Tuples with $|x_1| + \cdots + |x_n| = m$ and $x_n < 0$ correspond to elements of $S_{m-1,n-1}$ by dropping x_n. It follows that

$$\begin{aligned} &f(m,n) \\ &= f(m-1,n) + f(m,n-1) + f(m-1,n-1), \end{aligned}$$

so f satisfies a symmetric recurrence with symmetric boundary conditions $f(0,m) = f(n,0) = 1$. Hence f is symmetric.

Third solution: (by Greg Martin) As in the second solution, it is convenient to allow $f(m,0) = f(0,n) = 1$. Define the generating function

$$G(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m,n) x^m y^n.$$

As equalities of formal power series (or convergent series on, say, the region $|x|, |y| < \frac{1}{3}$), we have

$$\begin{aligned} G(x,y) &= \sum_{m \geq 0} \sum_{n \geq 0} x^m y^n \sum_{\substack{k_1, \dots, k_n \in \mathbb{Z} \\ |k_1| + \cdots + |k_n| \leq m}} 1 \\ &= \sum_{n \geq 0} y^n \sum_{k_1, \dots, k_n \in \mathbb{Z}} \sum_{m \geq |k_1| + \cdots + |k_n|} x^m \\ &= \sum_{n \geq 0} y^n \sum_{k_1, \dots, k_n \in \mathbb{Z}} \frac{x^{|k_1| + \cdots + |k_n|}}{1-x} \\ &= \frac{1}{1-x} \sum_{n \geq 0} y^n \left(\sum_{k \in \mathbb{Z}} x^{|k|} \right)^n \\ &= \frac{1}{1-x} \sum_{n \geq 0} y^n \left(\frac{1+x}{1-x} \right)^n \\ &= \frac{1}{1-x} \cdot \frac{1}{1 - y(1+x)/(1-x)} \\ &= \frac{1}{1-x-y-xy}. \end{aligned}$$

Since $G(x,y) = G(y,x)$, it follows that $f(m,n) = f(n,m)$ for all $m,n \geq 0$.

B5 **First solution:** Put $Q = x_1^2 + \cdots + x_n^2$. Since Q is homogeneous, P is divisible by Q if and only if each of the homogeneous components of P is divisible by Q. It is thus sufficient to solve the problem in case P itself is homogeneous, say of degree d.

Suppose that we have a factorization $P = Q^m R$ for some $m > 0$, where R is homogeneous of degree d and not divisible by Q; note that the homogeneity implies that

$$\sum_{i=1}^n x_i \frac{\partial R}{\partial x_i} = dR.$$

Write ∇^2 as shorthand for $\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$; then

$$\begin{aligned} 0 &= \nabla^2 P \\ &= 2mnQ^{m-1}R + Q^m \nabla^2 R + 2 \sum_{i=1}^n 2m x_i Q^{m-1} \frac{\partial R}{\partial x_i} \\ &= Q^m \nabla^2 R + (2mn + 4md) Q^{m-1} R. \end{aligned}$$

Since $m > 0$, this forces R to be divisible by Q, contradiction.

Second solution: (by Noam Elkies) Retain notation as in the first solution. Let P_d be the set of homogeneous

An alternate approach is to first rewrite $\sin x \sin x^2$ as $\frac{1}{2}(\cos(x^2 - x) - \cos(x^2 + x)$. Then

$$\int_0^B \cos(x^2 + x)\,dx = -\left.\frac{2x+1}{\sin(x^2+x)}\right|_0^B - \int_0^B \frac{2\sin(x^2+x)}{(2x+1)^2}\,dx$$

converges absolutely, and $\int_0^B \cos(x^2 - x)$ can be treated similarly.

A–5 Let a, b, c be the distances between the points. Then the area of the triangle with the three points as vertices is $abc/4r$. On the other hand, the area of a triangle whose vertices have integer coordinates is at least 1/2 (for example, by Pick's Theorem). Thus $abc/4r \geq 1/2$, and so

$$\max\{a, b, c\} \geq (abc)^{1/3} \geq (2r)^{1/3} > r^{1/3}.$$

A–6 Recall that if $f(x)$ is a polynomial with integer coefficients, then $m - n$ divides $f(m) - f(n)$ for any integers m and n. In particular, if we put $b_n = a_{n+1} - a_n$, then b_n divides b_{n+1} for all n. On the other hand, we are given that $a_0 = a_m = 0$, which implies that $a_1 = a_{m+1}$ and so $b_0 = b_m$. If $b_0 = 0$, then $a_0 = a_1 = \cdots = a_m$ and we are done. Otherwise, $|b_0| = |b_1| = |b_2| = \cdots$, so $b_n = \pm b_0$ for all n.

Now $b_0 + \cdots + b_{m-1} = a_m - a_0 = 0$, so half of the integers $b_0, \dots, b_{m-1}$ are positive and half are negative. In particular, there exists an integer $0 < k < m$ such that $b_{k-1} = -b_k$, which is to say, $a_{k-1} = a_{k+1}$. From this it follows that $a_n = a_{n+2}$ for all $n \geq k - 1$; in particular, for $m = n$, we have

$$a_0 = a_m = a_{m+2} = f(f(a_0)) = a_2.$$

B–1 Consider the seven triples (a, b, c) with $a, b, c \in \{0, 1\}$ not all zero. Notice that if r_j, s_j, t_j are not all even, then four of the sums $ar_j + bs_j + ct_j$ with $a, b, c \in \{0, 1\}$ are even and four are odd. Of course the sum with $a = b = c = 0$ is even, so at least four of the seven triples with a, b, c not all zero yield an odd sum. In other words, at least $4N$ of the tuples (a, b, c, j) yield odd sums. By the pigeonhole principle, there is a triple (a, b, c) for which at least $4N/7$ of the sums are odd.

B–2 Since $\gcd(m, n)$ is an integer linear combination of m and n, it follows that

$$\frac{gcd(m,n)}{n}\binom{n}{m}$$

is an integer linear combination of the integers

$$\frac{m}{n}\binom{n}{m} = \binom{n-1}{m-1} \text{ and } \frac{n}{n}\binom{n}{m} = \binom{n}{m}$$

and hence is itself an integer.

B–3 Put $f_k(t) = \frac{df^k}{dt^k}$. Recall Rolle's theorem: if $f(t)$ is differentiable, then between any two zeroes of $f(t)$ there exists a zero of $f'(t)$. This also applies when the zeroes are not all distinct: if f has a zero of multiplicity m at $t = x$, then f' has a zero of multiplicity at least $m - 1$ there.

Therefore, if $0 \leq a_0 \leq a_1 \leq \cdots \leq a_r < 1$ are the roots of f_k in $[0, 1)$, then f_{k+1} has a root in each of the intervals $(a_0, a_1), (a_1, a_2), \dots, (a_{r-1}, a_r)$, so long as we adopt the convention that the empty interval (t, t) actually contains the point t itself. There is also a root in the "wraparound" interval (a_r, a_0). Thus $N_{k+1} \geq N_k$.

Next, note that if we set $z = e^{2\pi i t}$; then

$$f_{4k}(t) = \frac{1}{2i}\sum_{j=1}^{N} j^{4k} a_j (z^j - z^{-j})$$

is equal to z^{-N} times a polynomial of degree $2N$. Hence as a function of z, it has at most $2N$ roots; therefore $f_k(t)$ has at most $2N$ roots in $[0, 1]$. That is, $N_k \leq 2N$ for all N.

To establish that $N_k \to 2N$, we make precise the observation that

$$f_k(t) = \sum_{j=1}^{N} j^{4k} a_j \sin(2\pi j t)$$

is dominated by the term with $j = N$. At the points $t = (2i + 1)/(2N)$ for $i = 0, 1, \dots, N - 1$, we have $N^{4k} a_N \sin(2\pi N t) = \pm N^{4k} a_N$. If k is chosen large enough so that

$$|a_N| N^{4k} > |a_1| 1^{4k} + \cdots + |a_{N-1}|(N-1)^{4k},$$

then $f_k((2i + 1)/2N)$ has the same sign as $a_N \sin(2\pi N a t)$, which is to say, the sequence $f_k(1/2N), f_k(3/2N), \dots$ alternates in sign. Thus between these points (again including the "wraparound" interval) we find $2N$ sign changes of f_k. Therefore $\lim_{k\to\infty} N_k = 2N$.

B–4 For t real and not a multiple of π, write $g(t) = \frac{f(\cos t)}{\sin t}$. Then $g(t + \pi) = g(t)$; furthermore, the given equation implies that

$$g(2t) = \frac{f(2\cos^2 t - 1)}{\sin(2t)} = \frac{2(\cos t) f(\cos t)}{\sin(2t)} = g(t).$$

In particular, for any integer n and k, we have

$$g(1 + n\pi/2^k) = g(2^k + n\pi) = g(2^k) = g(1).$$

Since f is continuous, g is continuous where it is defined; but the set $\{1 + n\pi/2^k | n, k \in \mathbb{Z}\}$ is dense in the reals, and so g must be constant on its domain. Since $g(-t) = -g(t)$ for all t, we must have $g(t) = 0$ when t is not a multiple of π. Hence $f(x) = 0$ for $x \in (-1, 1)$. Finally, setting $x = 0$ and $x = 1$ in the given equation yields $f(-1) = f(1) = 0$.

$k^2-k+1 \le n \le k^2+k$. Hence

$$\begin{aligned}
\sum_{n=1}^\infty \frac{2^{\langle n\rangle}+2^{-\langle n\rangle}}{2^n} &= \sum_{k=1}^\infty \sum_{n,\langle n\rangle=k} \frac{2^{\langle n\rangle}+2^{-\langle n\rangle}}{2^n} \\
&= \sum_{k=1}^\infty \sum_{n=k^2-k+1}^{k^2+k} \frac{2^k+2^{-k}}{2^n} \\
&= \sum_{k=1}^\infty (2^k+2^{-k})(2^{-k^2+k}-2^{-k^2-k}) \\
&= \sum_{k=1}^\infty (2^{-k(k-2)}-2^{-k(k+2)}) \\
&= \sum_{k=1}^\infty 2^{-k(k-2)} - \sum_{k=3}^\infty 2^{-k(k-2)} \\
&= 3.
\end{aligned}$$

Alternate solution: rewrite the sum as $\sum_{n=1}^\infty 2^{-(n+\langle n\rangle)} + \sum_{n=1}^\infty 2^{-(n-\langle n\rangle)}$. Note that $\langle n\rangle \neq \langle n+1\rangle$ if and only if $n=m^2+m$ for some m. Thus $n+\langle n\rangle$ and $n-\langle n\rangle$ each increase by 1 except at $n=m^2+m$, where the former skips from m^2+2m to m^2+2m+2 and the latter repeats the value m^2. Thus the sums are

$$\sum_{n=1}^\infty 2^{-n} - \sum_{m=1}^\infty 2^{-m^2} + \sum_{n=0}^\infty 2^{-n} + \sum_{m=1}^\infty 2^{-m^2} = 2+1 = 3.$$

B–4 For a rational number p/q expressed in lowest terms, define its *height* $H(p/q)$ to be $|p|+|q|$. Then for any $p/q \in S$ expressed in lowest terms, we have $H(f(p/q)) = |q^2-p^2|+|pq|$; since by assumption p and q are nonzero integers with $|p| \neq |q|$, we have

$$\begin{aligned}
H(f(p/q)) - H(p/q) &= |q^2-p^2|+|pq|-|p|-|q| \\
&\ge 3+|pq|-|p|-|q| \\
&= (|p|-1)(|q|-1)+2 \ge 2.
\end{aligned}$$

It follows that $f^{(n)}(S)$ consists solely of numbers of height strictly larger than $2n+2$, and hence

$$\cap_{n=1}^\infty f^{(n)}(S) = \emptyset.$$

Note: many choices for the height function are possible: one can take $H(p/q) = \max|p|, |q|$, or $H(p/q)$ equal to the total number of prime factors of p and q, and so on. The key properties of the height function are that on one hand, there are only finitely many rationals with height below any finite bound, and on the other hand, the height function is a sufficiently "algebraic" function of its argument that one can relate the heights of p/q and $f(p/q)$.

B–5 Note that $g(x)=g(y)$ implies that $g(g(x))=g(g(y))$ and hence $x=y$ from the given equation. That is, g is injective. Since g is also continuous, g is either strictly increasing or strictly decreasing. Moreover, g cannot tend to a finite limit L as $x \to +\infty$, or else we'd have $g(g(x))-ag(x)=bx$, with the left side bounded and the right side unbounded. Similarly, g cannot tend to a finite limit as $x \to -\infty$. Together with monotonicity, this yields that g is also surjective.

Pick x_0 arbitrary, and define x_n for all $n \in \mathbb{Z}$ recursively by $x_{n+1} = g(x_n)$ for $n>0$, and $x_{n-1} = g^{-1}(x_n)$ for $n<0$. Let $r_1 = (a+\sqrt{a^2+4b})/2$ and $r_2 = (a-\sqrt{a^2+4b})/2$ and r_2 be the roots of $x^2-ax-b=0$, so that $r_1>0>r_2$ and $1>|r_1|>|r_2|$. Then there exist $c_1,c_2 \in \mathbb{R}$ such that $x_n = c_1r_1^n + c_2r_2^n$ for all $n \in \mathbb{Z}$.

Suppose g is strictly increasing. If $c_2 \neq 0$ for some choice of x_0, then x_n is dominated by r_2^n for n sufficiently negative. But taking x_n and x_{n+2} for n sufficiently negative of the right parity, we get $0<x_n<x_{n+2}$ but $g(x_n)>g(x_{n+2})$, contradiction. Thus $c_2=0$; since $x_0=c_1$ and $x_1=c_1r_1$, we have $g(x)=r_1x$ for all x. Analogously, if g is strictly decreasing, then $c_2=0$ or else x_n is dominated by r_1^n for n sufficiently positive. But taking x_n and x_{n+2} for n sufficiently positive of the right parity, we get $0<x_{n+2}<x_n$ but $g(x_{n+2})<g(x_n)$, contradiction. Thus in that case, $g(x)=r_2x$ for all x.

B–6 Yes, there must exist infinitely many such n. Let S be the convex hull of the set of points (n,a_n) for $n \ge 0$. Geometrically, S is the intersection of all convex sets (or even all halfplanes) containing the points (n,a_n); algebraically, S is the set of points (x,y) which can be written as $c_1(n_1,a_{n_1})+\cdots+c_k(n_k,a_{n_k})$ for some $c_1,\dots,c_k$ which are nonnegative of sum 1.

We prove that for infinitely many n, (n,a_n) is a vertex on the upper boundary of S, and that these n satisfy the given condition. The condition that (n,a_n) is a vertex on the upper boundary of S is equivalent to the existence of a line passing through (n,a_n) with all other points of S below it. That is, there should exist $m>0$ such that

$$a_k < a_n + m(k-n) \qquad \forall k \ge 1. \tag{1}$$

We first show that $n=1$ satisfies (1). The condition $a_k/k \to 0$ as $k \to \infty$ implies that $(a_k-a_1)/(k-1) \to 0$ as well. Thus the set $\{(a_k-a_1)/(k-1)\}$ has an upper bound m, and now $a_k \le a_1+m(k-1)$, as desired.

Next, we show that given one n satisfying (1), there exists a larger one also satisfying (1). Again, the condition $a_k/k \to 0$ as $k \to \infty$ implies that $(a_k-a_n)/(k-n) \to 0$ as $k \to \infty$. Thus the sequence $\{(a_k-a_n)/(k-n)\}_{k>n}$ has a maximum element; suppose $k=r$ is the largest value of k that achieves this maximum, and put $m=(a_r-a_n)/(r-n)$. Then the line through (r,a_r) of slope m lies strictly above (k,a_k) for $k>r$ and passes through or lies above (k,a_k) for

$$f(e^{i\pi/3}) = \beta,$$
$$f(e^{-i\pi/3}) = \bar{\alpha} + 1 - \bar{\alpha}\beta,$$

where β is an arbitrary complex parameter.

(20th W.L. Putnam Competition, 1959)

537. Successively, we obtain

$$f(-1) = f\left(-\frac{1}{2}\right) = f\left(-\frac{1}{3}\right) = \cdots = \lim_{n\to\infty} f\left(-\frac{1}{n}\right) = f(0).$$

Hence $f(x) = f(0)$ for $x \in \{0, -1, -\frac{1}{2}, \ldots, -\frac{1}{n}, \ldots\}$.

If $x \neq 0, -1, \ldots, -\frac{1}{n}, \ldots$, replacing x by $\frac{x}{1+x}$ in the functional equation, we obtain

$$f\left(\frac{x}{1+x}\right) = f\left(\frac{\frac{x}{1+x}}{1 - \frac{x}{1+x}}\right) = f(x).$$

And this can be iterated to yield

$$f\left(\frac{x}{1+nx}\right) = f(x), \quad n = 1, 2, 3 \ldots.$$

Because f is continuous at 0 it follows that

$$f(x) = \lim_{n\to\infty} f\left(\frac{x}{1+nx}\right) = f(0).$$

This shows that only constant functions satisfy the functional equation.

538. Plugging in $x = t$, $y = 0$, $z = 0$ gives

$$f(t) + f(0) + f(t) \geq 3f(t),$$

or $f(0) \geq f(t)$ for all real numbers t. Plugging in $x = \frac{t}{2}$, $y = \frac{t}{2}$, $z = -\frac{t}{2}$ gives

$$f(t) + f(0) + f(0) \geq 3f(0),$$

or $f(t) \geq f(0)$ for all real numbers t. Hence $f(t) = f(0)$ for all t, so f must be constant. Conversely, any constant function f clearly satisfies the given condition.

(Russian Mathematical Olympiad, 2000)

539. No! In fact, we will prove a more general result.

Proposition. *Let S be a set and $g : S \to S$ a function that has exactly two fixed points $\{a, b\}$ and such that $g \circ g$ has exactly four fixed points $\{a, b, c, d\}$. Then there is no function $f : S \to S$ such that $g = f \circ f$.*

Proof. Let $g(c) = y$. Then $c = g(g(c)) = g(y)$; hence $y = g(c) = g(g(y))$. Thus y is a fixed point of $g \circ g$. If $y = a$, then $a = g(a) = g(y) = c$, leading to a contradiction. Similarly, $y = b$ forces $c = b$. If $y = c$, then $c = g(y) = g(c)$, so c is a fixed point of g, again a contradiction. It follows that $y = d$, i.e., $g(c) = d$, and similarly $g(d) = c$.

Suppose there is $f : S \to S$ such that $f \circ f = g$. Then $f \circ g = f \circ f \circ f = g \circ f$. Then $f(a) = f(g(a)) = g(f(a))$, so $f(a)$ is a fixed point of g. Examining case by case, we conclude that $f(\{a, b\}) \subset \{a, b\}$ and $f(\{a, b, c, d\}) \subset \{a, b, c, d\}$. Because $f \circ f = g$, the inclusions are, in fact, equalities.

Consider $f(c)$. If $f(c) = a$, then $f(a) = f(f(c)) = g(c) = d$, a contradiction since $f(a)$ is in $\{a, b\}$. Similarly, we rule out $f(c) = b$. Of course, c is not a fixed point of f, since it is not a fixed point of g. We are left with the only possibility $f(c) = d$. But then $f(d) = f(f(c)) = g(c) = d$, and this again cannot happen because d is not a fixed point of g. We conclude that such a function f cannot exist.

In the particular case of our problem, $g(x) = x^2 - 2$ has the fixed points -1 and 2, and $g(g(x)) = (x^2 - 2)^2 - 2$ has the fixed points $-1, 2, \frac{-1+\sqrt{5}}{2}$, and $\frac{-1-\sqrt{5}}{2}$. This completes the solution.

(B.J. Venkatachala, *Functional Equations: A Problem Solving Approach*, Prism Books PVT Ltd., 2002)

540. The standard approach is to substitute particular values for x and y. The solution found by the student S.P. Tungare does quite the opposite. It introduces an additional variable z. The solution proceeds as follows:

$$\begin{aligned}
&f(x+y+z)\\
&\quad = f(x)f(y+z) - c\sin x\sin(y+z)\\
&\quad = f(x)[f(y)f(z) - c\sin y\sin z] - c\sin x\sin y\cos z - c\sin x\cos y\sin z\\
&\quad = f(x)f(y)f(z) - cf(x)\sin y\sin z - c\sin x\sin y\cos z - c\sin x\cos y\sin z.
\end{aligned}$$

Because obviously $f(x + y + z) = f(y + x + z)$, it follows that we must have

$$\sin z[f(x)\sin y - f(y)\sin x] = \sin z[\cos x\sin y - \cos y\sin x].$$

Substitute $z = \frac{\pi}{2}$ to obtain

$$f(x)\sin y - f(y)\sin x = \cos x\sin y - \cos y\sin x.$$

For $x = \pi$ and y not an integer multiple of π, we obtain $\sin y[f(\pi) + 1] = 0$, and hence $f(\pi) = -1$.

Then, substituting in the original equation $x = y = \frac{\pi}{2}$ yields

$$f(\pi) = \left[f\left(\frac{\pi}{2}\right)\right] - c,$$

It follows that $f(x + x_n) - f(x)$ is a polynomial of degree $n - 2$ for all x_n. In particular, there exist polynomials $P_1(x)$ and $P_2(x)$ such that $f(x + 1) - f(x) = P_1(x)$, and $f(x + \sqrt{2}) - f(x) = P_2(x)$. Note that for any a, the linear map from the vector space of polynomials of degree at most $n - 1$ to the vector space of polynomials of degree at most $n - 2$, $P(x) \to P(x + a) - P(x)$, has kernel the one-dimensional space of constant polynomials (the only periodic polynomials). Because the first vector space has dimension n and the second has dimension $n - 1$, the map is onto. Hence there exist polynomials $Q_1(x)$ and $Q_2(x)$ of degree at most $n - 1$ such that

$$Q_1(x + 1) - Q_1(x) = P_1(x) = f(x + 1) - f(x),$$
$$Q_2(x + \sqrt{2}) - Q_2(x) = P_2(x) = f(x + \sqrt{2}) - f(x).$$

We deduce that the functions $f(x)-Q_1(x)$ and $f(x)-Q_2(x)$ are continuous and periodic, hence bounded. Their difference $Q_1(x)-Q_2(x)$ is a bounded polynomial, hence constant. Consequently, the function $f(x) - Q_1(x)$ is continuous and has the periods 1 and $\sqrt{2}$. Since the additive group generated by 1 and $\sqrt{2}$ is dense in $\mathbb{R}$, $f(x) - Q_1(x)$ is constant. This completes the induction.

That any polynomial of degree at most $n - 1$ with no constant term satisfies the functional equation also follows by induction on n. Indeed, the fact that f satisfies the equation is equivalent to the fact that g_{x_n} satisfies the equation. And g_{x_n} is a polynomial of degree $n - 2$.

(G. Dospinescu)

551. *First solution*: Assume that such functions do exist. Because $g \circ f$ is a bijection, f is one-to-one and g is onto. Since f is a one-to-one continuous function, it is monotonic, and because g is onto but $f \circ g$ is not, it follows that f maps $\mathbb{R}$ onto an interval I strictly included in $\mathbb{R}$. One of the endpoints of this interval is finite, call this endpoint a. Without loss of generality, we may assume that $I = (a, \infty)$. Then as $g \circ f$ is onto, $g(I) = \mathbb{R}$. This can happen only if $\limsup_{x\to\infty} g(x) = \infty$ and $\liminf_{x\to\infty} g(x) = -\infty$, which means that g oscillates in a neighborhood of infinity. But this is impossible because $f(g(x)) = x^2$ implies that g assumes each value at most twice. Hence the question has a negative answer; such functions do not exist.

Second solution: Since $g \circ f$ is a bijection, f is one-to-one and g is onto. Note that $f(g(0)) = 0$. Since g is onto, we can choose a and b with $g(a) = g(0) - 1$ and $g(b) = g(0) + 1$. Then $f(g(a)) = a^2 > 0$ and $f(g(b)) = b^2 > 0$. Let $c = \min(a^2, b^2)/2 > 0$. The intermediate value property guarantees that there is an $x_0 \in (g(a), g(0))$ with $f(x_0) = c$ and an $x_1 \in (g(0), g(b))$ with $f(x_1) = c$. This contradicts the fact that f is one-to-one. Hence no such functions can exist.

(R. Gelca, second solution by R. Stong)

552. The relation from the statement implies that f is injective, so it must be monotonic. Let us show that f is increasing. Assuming the existence of a decreasing solution f to