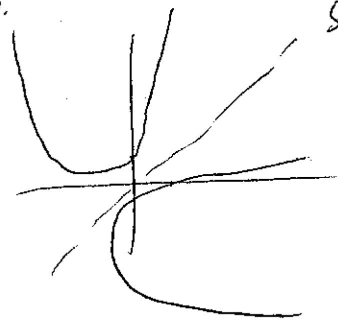
2007/A1

$y = \alpha x^2 + \alpha x + \frac{1}{24}$ tangent to $x = \alpha y^2 + \alpha y + \frac{1}{24}$.

If $\alpha \geq 0$.



Symmetric.

So tangent iff tangent to $y = x$

Need exactly one solution

$$x = \alpha x^2 + \alpha x + \frac{1}{24}.$$

$$0 = \alpha x^2 + (\alpha - 1)x + \frac{1}{24}$$

Need $B^2 - 4AC = 0$.

$$(\alpha - 1)^2 = 4\alpha \cdot \frac{1}{24} = \frac{1}{6}\alpha$$

$$\alpha^2 - 2\alpha + 1 = \frac{1}{6}\alpha.$$

$$\alpha^2 - \frac{13}{6}\alpha + 1 = 0$$

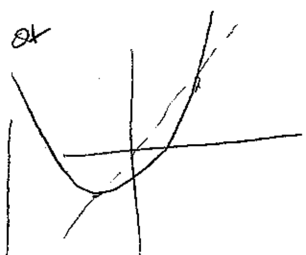
$$6\alpha^2 - 13\alpha + 6 = 0$$

3   -2
2   -3

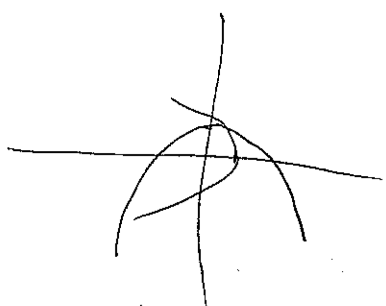
$$(3\alpha - 2)(2\alpha - 3) = 0.$$

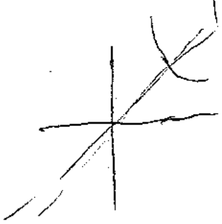
$$\alpha = \frac{2}{3} \quad \alpha = \frac{3}{2}.$$

or

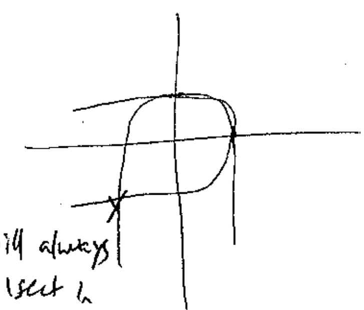


If $\alpha \leq 0$:

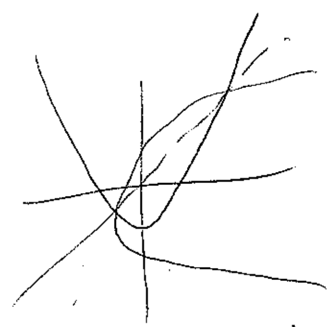




or:



will always isect in this quad



always opp sides of $y = x$.

2001 / #2

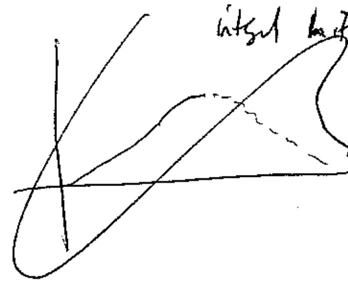
2010-09-27

2

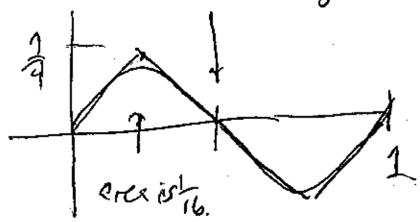
Given max der is ≤ 1

And ~~goes back to~~ integrates to 0

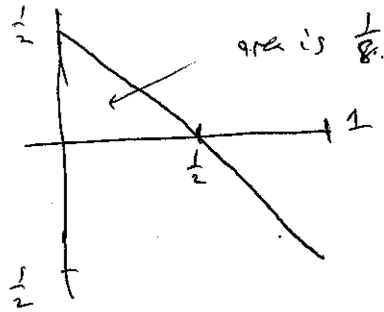
Need integral not to move much beyond $\frac{1}{8}$.



slope can't move
more than by ± 1

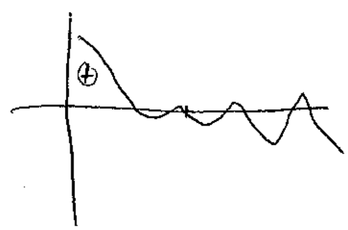


or



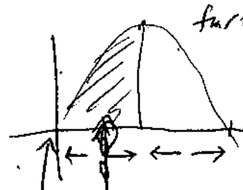
STS $+\int$ is bdd. Then can swap sign of f.

Will have max at critical point.



If your acceleration
is never more than ± 1,
and you end up where you started, what is farthest you could have gone?

$F(1) = F(0)$.
$|F''(t)| \leq 1$.



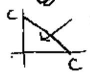
one is less than $\frac{1}{2}$ of time.

In half of time, starting from $F' = 0$ speed 0,
acc $= 1$,

how far can you go? The $F'' = +1$: it is $\frac{1}{2}t^2 \to \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8}$.

Rigorous: let max $\int_0^c f(x)\,dx$. Then $f(c) = 0$. By calculus

$|f(c) - f(x)| = \int_x^c |f'(t)|\,dt \leq M(c-x)$. IF $c \leq \frac{1}{2}$: then $M\int_0^c (c-x)\,dx$. ~~$\int f(x)\,dx$~~ $f(x) = \int_0^x f'(t)\,dt + f(0)$.



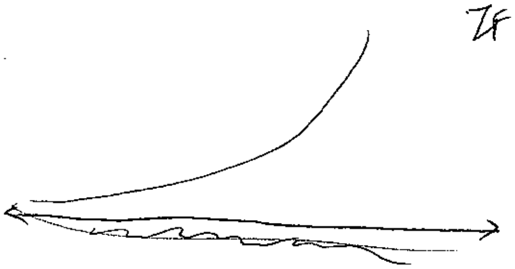
$\frac{1}{2}c^2 \leq \frac{1}{8}$

$f(c) = \int_0^c f'(t)\,dt =$

1998/A3

2010-09-28

3



If $f \leftrightarrow -f$ then keeps same sign

So what always ≥ 0.

And either incr / decr.

$g(x) = f(-x)$.

this swap preserves sign too. So WLOG incr

Sign patterns

f	f'	f''	f'''
+	+	+	−
+	+	−	+

only possibilities for contradiction

$f'' = e^{-x} + 100$

$f' = -e^{-x} + 100x$

$f = e^{-x}$

So

f'' is always ⊕, but decreasing.

So sample it at 0

Then left of 0, f'' is $\geq$ M_2 positive

At 0, f' is say M_1, positive.

~~To left of 0 f' only ...~~

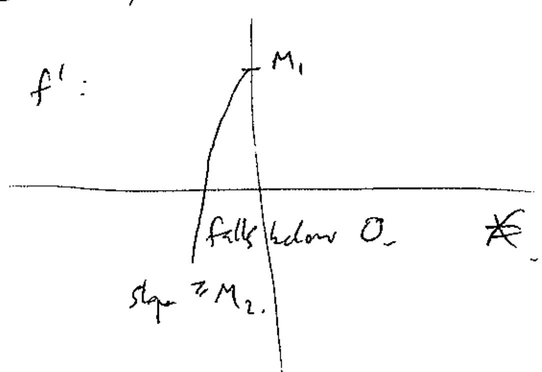
f' is positive at 0,

but to left of 0, f' is

So $f \longrightarrow$

same pattern

f':



C: every n numbers R has subset of size $\geq$ cn.

Given n #s. R. . No $b_1 + 2b_2 = 2b_3 + 2b_4$

show $\exists c$ s.t. $\exists$ subset of cn #s without that

2010-09-28
(4)

$F_0 = \ln x.$

$F_1(x) = \left[t\log t - t\right]_0^x = x\log x - x.$

$F_2(x) = \left[\log x \cdot \frac{x^2}{2} - \int\left(\frac{x}{2}+x\right)\right]_0^x = \frac{x^2}{2}\log x - \frac{1}{2}x^2\left(\frac{1}{2}+1\right).$

$F_3(x) = \left[\log x \cdot \frac{x^3}{3!} - \int \frac{x^2}{3!} + \frac{1}{2}x^2\left(\frac{1}{2}+1\right)\right].$

$= \frac{1}{2}x^2\left[\frac{1}{3}+\frac{1}{2}+1\right].$

$\downarrow$

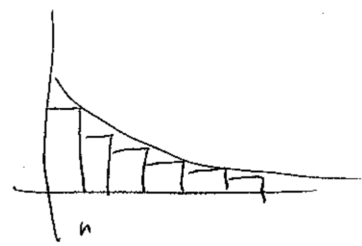
$\frac{x^3}{3!}\log x - \frac{1}{3!}x^3\left[\frac{1}{3}+\frac{1}{2}+1\right].$

$\vdots$

$F_n(x) = \frac{x^n}{n!}\log x - \frac{1}{n!}x^n\left[\frac{1}{n}+\frac{1}{n-1}+\dots+1\right].$

@ $x=1$: $-\frac{1}{n!}\underbrace{\left[\frac{1}{n}+\frac{1}{n-1}+\dots+1\right]}_{\to \log n}.$

$\to \log n$ since $\sum_{t=1}^{n}\frac{1}{t}$



$\log(n+1) = \int_1^{n+1}\frac{1}{t}\,dt \le \quad \le 1+\int_1^n \frac{1}{t}\,dt = 1+\log n.$

2010-09-28

1

$$|r^2+z^2+8| \le 6r.$$

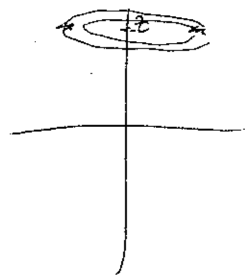
$$\Updownarrow$$

$$r^2+z^2+8 \le 6r.$$

$$z^2 \le 6r-r^2-8.$$

$$r^2-6r+8 \le -z^2$$

$$(r-4)(r-2) \le -z^2.$$



$dz \times (R^2-r^2)\pi.$

$\parallel$

$(R+r)(R-r)$

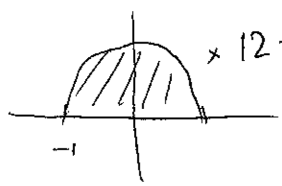
$$r^2-6r+(z^2+8) \le 0$$

$$r = \frac{6 \pm \sqrt{36-4z^2-32}}{2} = 3 \pm \sqrt{1-z^2}.$$ So only $z=-1..+1$.

$R+r=6$

$R-r=2\sqrt{1-z^2}$

$$V = \int_{-1}^{1} dz \times \pi \times \left[12\sqrt{1-z^2}\right].$$

=  $\times 12\pi = 12\pi \times \frac{\pi}{2} = 6\pi^2$

2002/A1 $\frac{1}{x^k-1}$ $\xrightarrow{\text{der } n}$ $\frac{P_n(x)}{(x^k-1)^{n+1}}$ find $P_n(1)$

$$(x^k-1)^{-1} \to (x^k-1)^{-2}(kx^{k-1})$$

$$\to -2(x^k-1)^{-3}(kx^{k-1}) + (x^k-1)^{-2}(k(k-1)x^{k-2})$$

$$= (x^k-1)^{-3}\Big[\underbrace{-2kx^{k-1}}_{-2k} + \underbrace{(x^k-1)(k(k-1)x^{k-2})}_{\text{eval at } 1 \to \text{ gone}}\Big].$$

$\to$

$$f(y) = \frac{1}{y}$$

$$g(x) = x^k-1. \quad g(1)=0. \quad g'(1)=k.$$

$$f(g(x)) = \frac{1}{x^k-1}.$$

$$f'(g(x)) = -\frac{1}{g^2}g'(x)$$

$$f''(g(x)) = \frac{2}{g^3}[g'(x)]^2 - \frac{1}{g^2}g''(x).$$

$$f'''(g(x)) = -\frac{3!}{g^4}[g'(x)]^3 + \frac{2}{g^3}g''(x)$$

+ Poly with g^2, g'.

$$P_n(1) = (-1)^n n!\, k^n.$$

A central feature of the Pittsburgh-based Simons Center will be an emphasis on interdisciplinary applications of the theory of computing. Computers have become a key component of essentially all research in the sciences and engineering, but the appreciation of fundamental computational theory by other disciplines has often lagged far behind their adoption of computers themselves. With computer models of complex physical phenomena, the analysis of very large data sets, and other computationally demanding tasks increasingly central to applied research there is a large and growing need for application domains to learn what the theory of computing has to offer them and, simultaneously, for the theory of computing to understand and address the special needs of each domain. This situation represents a tremendous opportunity for the field of computational theory to contribute to high-profile advances in science and technology and in the process to raise the profile and status of computational theory in the eyes of the public, funding agencies, future scientists, and the rest of the research community. Seizing this opportunity will, however, require a new model for how theory interacts with other domains inside and outside computer science, a model that a CMU-based Simons Center is ideally positioned to catalyze. Carnegie Mellon has long been a pioneer in interdisciplinary research that prides itself on the “low walls” between disciplines. The theory of computing at Carnegie Mellon is no exception, having long favored strong links between theory and practice. This philosophy was embodied in the university’s ALgorithmic ADaptation, Dissemination, and Integration (Aladdin) Center, which was founded with the goal of combining fundamental algorithmic theory with efforts to move this theory into practice. The process was facilitated by Aladdin’s highly successful Problem Oriented Exploration (PROBE) model, which used seed function to unite computational theorists and domain experts would unite around common applied problems. PROBES in such areas as authentication, privacy, genetics, economics, and parallel and distributed computing have simultaneously led to important theoretical advances and high-impact advances in their application domains. While the Aladdin Center itself is no longer funded, the practice is maintained by a close-knit group of theoretical computer scientists, including both Computer Science Department faculty who work in close collaboration with applied domain experts and computational theorists who have their primary appointments in other units (e.g., Mathematical Sciences, Biological Sciences, and the Tepper School of Business). This focus on dissemination has also continued through collaborations with industry through connections to local research laboratories of Google, Intel, and others and through the CMU-based Olympus Project, which facilitates export of university research into local startup companies. We intend for this culture of interdisciplinarity to infuse the new Simons Center, both informally through its interactions with CMU faculty and formally through the adoption of the PROBE model and related proven mechanisms for promoting theory in practice. The Simons Center can then serve as a platform for exporting this model of theory driven by and driving practice in other computation-heavy fields, building connections between theory and the sciences and in the process helping to revitalize the study of computational theory worldwide.

~~Ex~~ 1999/B2 2010-09-28 ①.

degree $\underline{n}$. $P(x) = Q(x)\ P''(x)$.
quad. 2^{nd} der.

$\underline{P}$ has ≥ 2 distinct roots
Need $\underline{n}$ distinct roots

~~Pf~~. Can we have repeated root?
Then $P(x) = (x-r)^2 \cdot R(x)$.

$$P'(x) = 2(x-r)R(x) + (x-r)^2 R'(x)$$

$$P''(x) = 2R(x) + 2(x-r)R'(x) + 2(x-r)R'(x) + (x-r)^2 R''(x).$$

$$= 2R(x) + 4(x-r)R'(x) + (x-r)^2 R''(x).$$

$$(x-r)^2 R(x) = Q(x)\left[2R(x) + 4(x-r)R'(x) + (x-r)^2 R''(x)\right]$$

$\Rightarrow (x-r) \mid Q(x)R(x)$.

If $(x-r) \mid Q$, then still $(x-r) \mid \frac{Q}{(x-r)} R$. $\Rightarrow$ again $(x-r) \mid Q$

So $Q = c(x-r)^2$.

$$\cancel{(x-r)^2} R(x) = c\cancel{(x-r)^2}\left[2R(x) + 4(x-r)R'(x) + (x-r)^2 R''(x)\right],$$

$\Rightarrow$ either $c = \frac{1}{2}$ or $(x-r) \mid R(x)$ again.

$$0 = 4(x-r)R'(x) + (x-r)^2 R''(x).$$

$$0 = 4R'(x) + (x-r)R''(x).$$

$$R'(x) = -\tfrac{1}{4}(x-r)R''(x).$$ ~~again~~

$$R''(x) = -\tfrac{1}{4}R''(x) - \tfrac{1}{4}(x-r)R'''(x)$$

$P(y) = Q(y)\ P''(y)$ and 2 roots ~~at~~ $y=0$ $\Rightarrow P(y) = y^2 \cdots$

~~Let P(y~~ if $P(y) = cy^2 \cdot P''(y)$.

Let $p(y) = a_t y^t + \cdots + a_n y^n = y^t R(y)$ $P'(y) = t y^{t-1} R(y) + y^t R'(y)$

$P''(y) =$ $P''(y) = t(t-1)y^{t-2}R(y) +$

2010-09-28
2

Say root is at $y=0$.

Then $P(y) = a_t y^t + \dots + a_n y^n = y^t R(y)$.

$P'(y) = t y^{t-1} R(y) + y^t R'(y)$

$P''(y) = t(t-1) y^{t-2} R(y) + t y^{t-1} R'(y) + t y^{t-1} R'(y) + y^t R''(y)$

$= t(t-1) y^{t-2} R(y) + 2t y^{t-1} R'(y) + y^t R''(y)$.

Yet $P(y) = Q(y) P''(y)$.

$\cancel{E = Q(y)}$

$$\underbrace{y^t}_{y^{t-1}} R(y) = Q(y) \cdot \left[t(t-1) y^{t-2} R(y) + \underbrace{2t y^{t-1} R'(y)}_{y^{t-1}} + \underbrace{y^t R''(y)}_{y^{t-1}} \right].$$

$\Rightarrow y \mid Q(y) R(y)$, and not in R, so $y \mid Q$

THEN

y^t y^{t} y^{t}

So $y^t \mid Q y^{t-2} R \Rightarrow y^2 \mid Q \Rightarrow Q = c y^2$.

So: $P''(y) = \frac{1}{cy^2} P(y)$: shifting coeffs down

$$a_t (t)(t-1) y^{t-2} + a_{t+1} (t+1)(t) y^{t-1} + \vdots \quad \parallel \quad \frac{1}{c} \cdot \left(a_t y^{t-2} + a_{t+1} y^{t-1} + \dots + a_n y^{n-2} \right)$$

Compare coeffs... can't have same $\le$ unless only one term

And we said that first term was $a_t y^t$.

$\Rightarrow$ Poly is $c y^n$.

$\square$.