SZEMEREDI 4-TERMS AP

N always prime, not 2 or 3.

Let $f_i : \mathbb{Z}_N \to [-1,1]$. Define:

$$\Lambda_3(f_1,f_2,f_3) = \mathbb{E}_{x,d}\, f_1(x) f_2(x+d) f_3(x+2d)$$

$$\Lambda_4(f_1,f_2,f_3,f_4) = \mathbb{E}_{x,d}\, f_1(x) f_2(x+d) f_3(x+2d) f_4(x+3d)$$

Note: $\Lambda_3(A_1,A_2,A_3)$ counts 3-AP's with first term from A_1, etc, but mult by $\frac{1}{N^2}$.

Suppose $A \subseteq \mathbb{Z}_N$, with density α $(|A| = \alpha N)$

(D) $f_A = 1_A - \alpha$ "balanced function of A".

BALANCED FCN DECOMP Say $|A_i| = \alpha_i N$.

Then $\Lambda_3(A_1,A_2,A_3) = \alpha_1\alpha_2\alpha_3 + (7 \text{ other terms})$

$\Lambda_4(A_1,A_2,A_3,A_4) = \alpha_1\alpha_2\alpha_3\alpha_4 + (15 \text{ other terms})$.

"main terms" measure "non-uniformity"

Each of "other terms" is of form $\Lambda_3(g_1,g_2,g_3)$ where some g_i is the balanced function f_{A_i}.

Consider special case where A_i equal.

Def. "Uniformity along 3-term progressions".

Let $|A| = \alpha N$, $f_A = 1_A - \alpha$. Then "A exhibits δ-uniformity along 3-term AP's" if whenever we have $g_1, g_2, g_3 : \mathbb{Z}_N \to [-1,1]$, at least one of which is equal to f_A, then

$$|\Lambda_3(g_1,g_2,g_3)| \le \delta.$$

(Define unif. along 4-term AP similarly)

REMARK. Not obvious that $\exists$ sets that are unif along progressions

COROLLARY. $|A| = \alpha N$. If A is δ-unif along 3-AP, then $|\Lambda_3(A,A,A) - \alpha^3| \le 7\delta$, and sim. for 4-AP.

QUESTION. Suppose A is not δ-unif. along 3- or 4-AP. What can we say about A?

(Equiv: Find sufficient condition for δ-unif).

Here, we define $\hat{f}(r) = \mathbb{E}_x f(x) e^{\frac{2\pi i}{N} x r}$. So normalization already in.

* FOR 3-AP, ATTEMPT I. Say A is not δ-unif along 3-AP. Then $\|\hat{f}_A\|_\infty \ge \delta$.

(P). Say $|\Lambda_3(g_1,g_2,f_A)| \ge \delta$ (case when say $g_2 = f_A$ is similar).

$$\Lambda_3(g_1,g_2,f_A) = N\mathbb{E}_{x,y,z,r} e^{\frac{2\pi i}{N}(x-2y+z)r} f_1(x) f_2(y) f_3(z) = N\mathbb{E}_r \left(\mathbb{E}_x f_1(x) e^{\frac{2\pi i}{N} r x}\right)\left(\mathbb{E}_y f_2(y) e^{\frac{2\pi i}{N}(-2yr)}\right)\left(\mathbb{E}_z f_3(z) e^{\frac{2\pi i}{N} zr}\right)$$

Since we switched to $\mathbb{E}_z$ instead of $\sum_z$. $\mathbb{E}_r$ is already $\begin{cases} 0 & \text{if } x-2y+z \ne 0 \\ 1 & \text{if } x+z = 2y \end{cases}$

$$= N\mathbb{E}_r\, \hat{g}_1(r)\, \hat{g}_2(-2r)\, \hat{f}_A(r).$$

$$= \sum_r \hat{g}_1(r)\, \hat{g}_2(-2r)\, \hat{f}_A(r).$$

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↓

(B)

Hence $\delta \le \left| \sum_r \hat{g}_1(r)\, \hat{g}_2(-2r)\, \hat{f}_A(r) \right| \le \|\hat{f}_A\|_\infty \cdot \|\hat{g}_1\|_2 \cdot \|\hat{g}_2\|_2 = \|\hat{f}_A\|_\infty \cdot \sqrt{E_x g_1^2(x)} \cdot \sqrt{E_x g_2^2(x)}$.

Pull $\hat{f}_A$ out, then use Cauchy-Schwarz

$\le \|\hat{f}_A\|_\infty$ $\square$

Parseval.

Clean proof, but not generalizable.

"One way to generalize an argument is to first try and find a more longwinded, less natural approach and try and generalize that.":

*** FOR 3AP, ARGUMENT II** Say A is not δ-unif along 3-AP. Then $\|\hat{f}_A\|_\infty \gtrsim \delta^2$.

(P) Reparametrization: $\Lambda_3(g_1, g_2, f_A) = E_{y_1,y_2}\, g_1(-y_1)\, g_2(\tfrac{1}{2}y_2)\, f_A(y_1+y_2)$

this is inverse of 2 in $\mathbb{Z}_N$, not $\tfrac{1}{2} \in \mathbb{Q}$.

Cauchy-Schwarz: $|\Lambda_3|^2 \le \Big(E_{y_2} \underbrace{|g_2(\tfrac{1}{2}y_2)|^2}_{\text{we assumed } |g_2| \le 1.}\Big) \cdot \Big(E_{y_2} \big|E_{y_1} g_1(-y_1)\, f_A(y_1+y_2)\big|^2\Big)$.

$$\le E_{y_2} \cdot \big|E_{y_1} g_1(-y_1)\, f_A(y_1+y_2)\big|^2.$$

$$= E_{y_2} \cdot E_{y_1,y_1'}\, g_1(-y_1)\, f_A(y_1+y_2)\, \overline{g_1(-y_1')\, f_A(y_1'+y_2)}$$

Cauchy-Schwarz $|\Lambda_3|^4 \le \Big(E_{y_1,y_1'} \underbrace{\big|g_1(-y_1)\,\overline{g_1(-y_1')}\big|^2}_{|g_i| \le 1.}\Big) \cdot \Big(E_{y_1,y_1'} \big|E_{y_2} f_A(y_1+y_2)\,\overline{f_A(y_1'+y_2)}\big|^2\Big)$

$$\le E_{y_1,y_1'} \left[E_{y_2} f_A(y_1+y_2)\,\overline{f_A(y_1'+y_2)}\right]^2.$$

$$= E_{y_1,y_1'}\, E_{y_2,y_2'}\, f_A(y_1+y_2)\,\overline{f_A(y_1'+y_2)}\;\overline{f_A(y_1+y_2')}\, f_A(y_1'+y_2')$$

$=: \|f_A\|_{U^2}^4$ defined to be Gowers U^2 norm

Reparametrization of Gowers norm:

$$\|f\|_{U^2}^4 = E_{x,h_1,h_2}\, f(x)\,\overline{f(x+h_1)}\;\overline{f(x+h_2)}\, f(x+h_1+h_2)$$

$$= E_{x,h_1,h_2}\, \Delta(f;h_1,h_2)(x)$$

(D) $\Delta(f;h) = f(x)\overline{f(x-h)}$ "Derivative?".

$\Delta(f;h_1,h_2) = \Delta(\Delta(f;h_1);h_2)$.

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Get used to Δ notation.

$$\|f\|_{U^2}^4 = E_{x,h_1,h_2}\, \Delta(f;h_1)(x)\cdot\overline{\Delta(f;h_1)(x+h_2)}$$
$$= E_h\, E_{x,y}\, \Delta(f;h)(x)\,\overline{\Delta(f;h)(y)}$$
$$= E_h \cdot E_x |\Delta(f;h)(x)|^2$$
(Parseval) $$= E_h \sum_r |\hat{\Delta}(f;h)(r)|^2$$

$$\hat{\Delta}(f;h)(r) = E_x\, \Delta(f;h)(x)\, e^{\frac{2\pi i}{N}rx}.$$
$$\cancel{= E_x f(x)\overline{f(x-h)}\, e^{\frac{2\pi i}{N}rx}}$$
$$\cancel{= E_x f(x)\, e^{\frac{2\pi i}{N}rx}}$$

def. by $f*g(x) = E_y f(x) g(x-y)$

$$\|f*f\|_2^2 = E_x\, f*f(x)\cdot\overline{f*f(x)}$$
$$= E_{x,y,z}\, f(y) f(x-y)\, \overline{f(z) f(x-z)}.$$

$h_1 = z-y.$
$h_2 = x-z-y.$
$y+h_1+h_2 = x-y.$

$$= E_{y,h_1,h_2}\, f(y) f(y+h_1+h_2)\,\overline{f(y+h_1) f(y+h_2)} = \|f\|_{U^2}^4.$$

Yet $\|f*f\|_2^2 = \|\widehat{f*f}\|_2^2 =: \sum_r |\widehat{f*f}(r)|^2 = \sum_r |\hat{f}(r)|^4 =: \|\hat{f}\|_4^4.$

Hence $\|f\|_{U^2} = \|\hat{f}\|_4.$

In particular, if we assumed $|\Lambda_3| \geq \delta$, then $\|f_A\|_{U^2} \geq \delta^4$
$\Rightarrow \|\hat{f}_A\|_4 \geq \delta^4.$

Yet then $\delta^4 \leq \|\hat{f}_A\|_4^4 \leq \|\hat{f}_A\|_\infty^2 \cdot \|\hat{f}_A\|_2^2 \underset{\uparrow \text{ Parseval}}{=} \|f_A\|_\infty^2 \cdot \underbrace{\|\hat{f}_A\|_2^2}_{\text{assumed is } g_3,\text{ and } |g_3|\leq 1} \leq \|f_A\|_\infty^2$

$$\Rightarrow \boxed{\|f_A\|_\infty \geq \delta^2.} \qquad \square.$$

Note: in this proof, we did not need circle method, which may be hard to generalize to 4-term AP, since 4-AP is 2 simultaneous eqns, not just $x-2y+z=0$ (single eqn)

DIVISION OF LABOR

(Generalized von Neumann thm): "Λ_3" controlled by Gowers U^2-norm:

$\forall f_i : \mathbb{Z}_N \to \mathbb{D},\quad |\Lambda_3(f_1,f_2,f_3)| \leq \inf \|f_i\|_{U^2}$

(Gowers Inverse thm): if Gowers U^2 norm is large, then $\exists$ large Fourier coeff.

$\forall f : \mathbb{Z}_N \to \mathbb{D},\quad \|f\|_{U^2} \geq \delta \Rightarrow \|\hat{f}\|_\infty \geq \delta^2.$

~~2004~~ 2007-04-09

(D)

GEN. VON-NEUMANN THM FOR 4-AP, let $f_1, \dots f_4 : \mathbb{Z}_N \to [-1,1]$.

Then $|\Lambda_4(f_1, \dots, f_4)| \le \inf \|f_i\|_{U^3}$,

where $\|f\|_{U^3}^8 := E_{x,h_1,h_2,h_3}\, \Delta(x; h_1, h_2, h_3)$.

= "averages over 3-dimensional boxes".

$$= E_{y_1,y_2,y_3,y_1',y_2',y_3'}\, \overline{f(y_1+y_2+y_3)}\, f(y_1+y_2+y_3')\, f(y_1+y_2'+y_3)\, \overline{f(y_1+y_2'+y_3')} \times$$
$$\times\, f(y_1'+y_2+y_3)\, \overline{f(y_1'+y_2+y_3')}\, \overline{f(y_1'+y_2'+y_3)}\, f(y_1'+y_2'+y_3').$$

(P). Again, we find "suitable reparametrization", and use Cauchy-Schwarz 3 times to get rid of the g_i.

Reparametrize: $\Lambda_4 = E_{y_1,y_2,y_3}\, f_1(-\tfrac{1}{2}y_2 - 2y_3)\, f_2(\tfrac{1}{3}y_1 - y_3)\, f_3(\underbrace{\tfrac{2}{3}y_1 + \tfrac{1}{2}y_2}_{+\frac{1}{2}y_2})\, f_4(y_1+y_2+y_3)$

For this section, let $B(\cdot)$ be any fn bdd by 1.

Cauchy-Schwarz: ~~$|E_x E_y B(x) f(x,y)|^2 \le E_x$~~

Let X, Y be sets of variables, and if $X = \{x_1, x_2\}$, let E_X mean E_{x_1,x_2}

$$\left|E_X E_Y B(X) f(X,Y)\right|^2 \le \left(E_X |B(X)|^2\right)\left(E_X |E_Y f(X,Y)|^2\right)$$

$$\le E_{X, Y^{(1)}, Y^{(2)}}\, f(X, Y^{(1)})\, \overline{f(X, Y^{(2)})}.$$

2 copies of variable sets.

Apply 3 times

1. $X = \{y_2, y_3\}$, $Y = \{y_1\}$, $B = f_1$
2. $X = \{y_1^{(1)}, y_1^{(2)}, y_3\}$, $Y = \{y_2\}$, B = everything involving f_2
3. $X = \{y_1^{(1)}, y_1^{(2)}, y_2^{(1)}, y_2^{(2)}\}$, $Y = \{y_3\}$, B = everything with f_3.

Eliminates f_1, f_2, f_3

$\Rightarrow |\Lambda_4|^8 \le \|f_4\|_{U^3}^8$. ~~Others~~ Bound by $\|f_i\|_{U^3}^8$ with $i \ne 4$ similar.

So if

Counting 4-APs.

Let $A_1, \dots, A_4 \subseteq \mathbb{Z}_N$, with densities $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Say A_3, A_4 ~~are δ-unif along 4-AP~~ satisfy $\|f_{A_3}\|_{U^3}, \|f_{A_4}\|_{U^3} \le \delta$. (So δ-unif along 4-AP) Then if $\delta \le \frac{\alpha_1\alpha_2\alpha_3\alpha_4}{6}$,

$A_1 \times A_2 \times A_3 \times A_4$ contains $\geq \frac{1}{2}\alpha_1\alpha_2\alpha_3\alpha_4 N^2$ $\mathbb{Z}_N$-AP's.

(P). ~~$\Lambda(f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}) =$~~

$$\# = \sum_{x,d} A_1(x) A_2(x+d) A_3(x+2d) A_4(x+3d)$$

f_3, f_4 are Balanced fns of A_3, A_4.

$$= \sum_{x,d} A_1(x) A_2(x+d)\big(f_3(x+2d) + \alpha_3\big)\big(f_4(x+3d) + \alpha_4\big).$$

$$= \underbrace{\sum_{x,d} A_1(x) A_2(x+d)}_{=\sum_{x,y} A_1(x) A_2(y) = \alpha_1\alpha_2 N^2} \alpha_3\alpha_4 + (3 \text{ other terms})$$

all 3 have f_3 or f_4, so by Gen. Von-Neumann,

$$\Lambda(A_1, A_2, \underset{f_3 \text{ or } f_4}{*, *}) \le \|f_3\|_{U^3} \le \delta.$$

$$\geq N^2 \alpha_1\alpha_2\alpha_3\alpha_4 - 3 \cdot \delta N^2.$$

$$\geq N^2 \alpha_1\alpha_2\alpha_3\alpha_4 \cdot \frac{1}{2}.$$

□

Corollary 7.6 $|A| = \alpha N$, say $\|f_A\|_{U^3} \le \delta$. Then either find 4-AP, or $\exists$ subprogression where A has density $\geq \frac{9}{8}\alpha$.

↳ just need $\delta \le \frac{1}{6}\left(\frac{\alpha}{10}\right)\left(\frac{\alpha}{10}\right)\alpha\alpha = \frac{\alpha^4}{600}$.

(P). Let $A_1 = A_2 = A \cap \left[\frac{2N}{5}, \frac{3N}{5}\right]$ (middle fifth)

$A_3 = A_4 = A$.

If $|A_1| \le \frac{\alpha N}{10}$, then either $A \cap [0, \frac{2}{5}N]$ or $A \cap [\frac{3}{5}N, N)$ has size $\geq \frac{9\alpha N}{20} \Rightarrow$ density $\geq \frac{9\alpha N}{20} \times \frac{5}{2N}$
$= \frac{9}{8}\alpha$.

So ~~suppose~~ we may assume $|A_1| \geq \frac{\alpha N}{10} \Rightarrow$ density $\alpha_1 \geq \frac{\alpha}{10}$ (density is with respect to $\mathbb{Z}_N$)

Previous statement then implies $\exists \geq \frac{1}{2}\left(\frac{\alpha}{10}\right)\left(\frac{\alpha}{10}\right)(\alpha)(\alpha)N^2$ 4-AP's, so done.

□

SZEMERÉDI 4-AP IF NOT QUADRATICALLY UNIFORM

Recall def of Quadratically Uniform was $\|f\|_{U^3}^8 = \mathbb{E}_{x,h_1,h_2,h_3} \Delta(f;h_1,h_2,h_3)(x)$.

Since we do not want to keep 8^{th} roots everywhere, we ~~use $\mathbb{E}$~~ now define: QUADRATICALLY α-UNIFORM: $\|f\|_{U^3}^8 \le \alpha$.

Many ways to write this: (now we stop using Expectation)

$$\underbrace{\sum_{x,h_1,h_2,h_3} \Delta(f;h_1,h_2,h_3)(x)} \le \alpha N^4 \iff \|f\|_{U^3}^8 \le \alpha.$$

$$\sum_{x,k,l,m} \Delta(f;k,l,m)(x) \underset{\text{def of } \Delta}{=} \sum_{x,k,l,m} \Delta(f;k)(x)\, \overline{\Delta(f;k)(x-l)}\, \overline{\Delta(f;k)(x-m)}\, \Delta(f;k)(x-l-m)$$

~~letting~~ Let $F_k = \Delta(f;k)$

$$= \sum_k \sum_{x,l,m} F_k(x)\, \overline{F_k(x-l)}\, \overline{F_k(x-m)}\, F_k(x-l-m)$$

by argument on 2007-04-09 (C)

$$\to = \sum_k \sum_x F_k * F_k(x) \cdot \overline{F_k * F_k(x)}$$

$$= \sum_k \sum_x \left| F_k * F_k(x) \right|^2$$

$$= \frac{1}{N} \sum_k \sum_r \left| \widehat{F_k * F_k}(r) \right|^2 \quad \longleftarrow \text{Now we use } \hat{f}(r) = \sum_x f(x) e^{\frac{2\pi i r x}{N}}.$$

$$= \frac{1}{N} \sum_k \sum_r \left| \hat{F}_k(r) \right|^4.$$

$$\boxed{\Rightarrow \text{We may say } \underline{\text{QUADRATIC } \alpha\text{-UNIF}}: \ \sum_{k,r} \left| \hat{\Delta}(f;k)(r) \right|^4 \le \alpha N^5.}$$

GOAL: If A is NOT Quadratically α-unif, then A intersects a $\mathbb{Z}$-AP of size $\ge N^d$ such that density is $\ge \delta + \varepsilon$ and ε, d depend only on α and δ.

density of A — uniformity — density of A

Lemma 7.7 f not quad α-unif. Then $\exists B \subseteq \mathbb{Z}_N$ of size $\geq \frac{\alpha N}{2}$, and function $\phi: B \to \mathbb{Z}_N$ s.t.

$$\sum_{k \in B} |\hat{\Delta}(f;k)(\phi(k))|^2 \geq \left(\frac{\alpha}{2}\right)^2 N^3.$$

(P) Not quad α-unif: $\sum_{k,r} |\hat{\Delta}(f;k)(r)|^4 > \alpha N^5$.

(1) We want to say that for $\geq \frac{\alpha N}{2}$ values of k, $\sum_r |\hat{\Delta}(f;k)(r)|^4 > \frac{\alpha N^4}{2}$.

(2) Then we will show that for any such k, $\|\hat{\Delta}(f;k)(r)\|_\infty^2 > \frac{\alpha N^2}{2}$.

Summing over the $\geq \frac{\alpha N}{2}$ values of k will give us the result.

Pf(1). First we want the trivial bound $\sum_r |\hat{\Delta}(f;k)(r)|^4 = \sum_r |\widehat{\Delta(f;k) * \Delta(f;k)}(r)|^2$

$$= N \cdot \sum_x |\Delta(f;k) * \Delta(f;k)(x)|^2$$

ordinary convolution, no expectation, so $f*g(x) = \sum_y f(y)g(x-y)$.

So since $\|\Delta(f;k)\|_\infty \leq \|f\|_\infty^2 \leq 1$, the convolution is bdd by N.

$$\leq N \cdot \sum_x N^2 = N^4.$$

Yet worst case is having values either just at $\frac{\alpha}{2}N^4$ or at N^4.

Max. contribution of the $\sum_r |\hat{\Delta}(f;k)(r)|^4$ that are $\frac{\alpha}{2}N^4$ is $\leq N \cdot \left(\frac{\alpha}{2}N^4\right)$,

so the other big guys need to contribute total of $\geq \frac{\alpha}{2}N^5$, and dividing by N^4 gives that their number must be $\geq \frac{\alpha}{2}N$, as claimed.

Pf(2). Say we have k s.t. $\sum_r |\hat{\Delta}(f;k)(r)|^4 > \frac{\alpha N^4}{2}$.

But then $\frac{\alpha N^4}{2} < \sum_r |\hat{\Delta}(f;k)(r)|^4 \leq \|\cdot\|_\infty^2 \sum_r |\hat{\Delta}(f;k)(r)|^2 = \|\cdot\|_\infty^2 N \underbrace{\sum_x |\Delta(f;k)(x)|^2}_{\leq \|f\|_\infty^2 \leq 1} \leq \|\cdot\|_\infty^2 N^2$

$\Rightarrow \|\cdot\|_\infty^2 > \frac{\alpha N^2}{2}$, as desired. $\square$.

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A

LEMMA 7.8 Say $f: \mathbb{Z}_N \to [-1,1]$, $B \subseteq \mathbb{Z}_N$, and $\phi: B \to \mathbb{Z}_N$ s.t. $\sum_{k \in B} |\hat{\Delta}(f;k)(\phi(k))|^2 \geq \alpha N^3$.

THEN: $\exists \geq \alpha^4 N^3$ quadruples $(a,b,c,d) \in B^4$ s.t. $\begin{cases} a+b = c+d \\ \phi(a)+\phi(b) = \phi(c)+\phi(d) \end{cases}$

~~First need LEMMA 7.1 $g: \mathbb{Z}_N \to [-1,1]$. Then g is α-unif (def: $\sum_r |\hat{g}(r)|^4 \leq \alpha N^4$)~~

~~if $\forall F: \mathbb{Z}_N \to \mathbb{C}$, $\sum_s |\sum_k F(s-k) g(k)|^2 \leq \sqrt{\alpha} N^2 \|F\|_2^2$~~

Need Lem 7.1 (adjusted): $\sum_r |\hat{g}(r)|^4 \geq \left(\dfrac{\sum_s |\sum_k F(s-k) g(k)|^2}{N} \right)^2$

if $|F| \leq 1$, then

Pf Suff. show: $\sum_s |\sum_k F(s-k) g(k)|^2 \leq \sqrt{\sum_r |\hat{g}(r)|^4 \cdot N^2}$.

Well, $\sum_s |(F*g)(s)|^2 = \frac{1}{N} \sum_r |\widehat{(F*g)}(r)|^2 = \frac{1}{N} \sum_r |\hat{F}(r)\hat{g}(r)|^2$

$= \frac{1}{N} \sum_r |\hat{F}(r)|^2 \cdot |\hat{g}(r)|^2 \underset{C\text{-}S}{\leq} \frac{1}{N} \sqrt{\sum_r |\hat{F}(r)|^4 \sum_r |\hat{g}(r)|^4}$

Yet $\sum_r |\hat{F}(r)|^4 = \sum_r |\widehat{F*F}(r)|^2 = N \cdot \sum_x |\underbrace{(F*F)(x)}_{\leq N}|^2 \leq N^4$, which is what we wanted

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Pf L7.8 Rewrite: $\sum_{k \in B} |\hat{\Delta}(f;k)(\phi(k))|^2 = \sum_{k \in B} \sum_{s,t} \Delta(f;k)(s)\, \omega^{\phi(k)s}\, \overline{\Delta(f,k)(t)\, \omega^{\phi(k)t}}$

$u = s-t$. $= \sum_{k \in B} \sum_{s,u} f(s)\overline{f(s-k)}\;\overline{f(s-u)}f(s-u-k)\; \omega^{\phi(k)u}$

$\alpha N^3 \leq \sum_{u,s} \left| \sum_{k} \underbrace{\overline{f(s-k)} f(s-k-u)}_{\text{This is } F_u(s-k)}\; \underbrace{B(k)\omega^{\phi(k)u}}_{\text{This is } g_u(k)} \right|$ — indicator fn of B.

$\Rightarrow \dfrac{(\alpha N^3)^2}{N^2} = \alpha^2 N^4$. $\leq \sum_u \sum_s \left| \sum_{k} F_u(s-k)\, g_u(k) \right|^2$

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B

↓ So let $N^3\gamma(u) = \sum_s \left|\sum_{k\in B} F_u(s-k)\, g_u(k)\right|^2$.

Our prev. ineq says $\sum_u \gamma(u) \geq \alpha^2 N$ $\Rightarrow$ $\sum_u \gamma(u)^2 \geq \frac{(\alpha^2 N)^2}{N} = \alpha^4 N$

But (L7.1) $\Rightarrow \sum_r |\hat{g}_u(r)|^4 \geq \gamma(u)^2 N^4$.

$$\gamma(u)^2 N^4 \leq \left\{ \begin{aligned} \sum_r |\hat{g}_u(r)|^4 &= \sum_r \left|\sum_k B(k)\,\omega^{\phi(k)u}\,\omega^{rk}\right|^4 \\ &= \sum_r \sum_{a,b,c,d\in B} \omega^{(\phi(a)+\phi(b)-\phi(c)-\phi(d))u}\,\omega^{r(a+b-c-d)} \end{aligned}\right.$$

Sum over $\underline{u}$:

$$\alpha^4 N^5 \leq \sum_u \gamma(u)^2 N^4 \leq \sum_{r,u} \sum_{a,b,c,d\in B} \omega^{(\phi - \phi)u}\,\omega^{r(--)}$$

$$= N^2 \times \#\text{of quadruples with } \begin{cases} a+b = c+d \\ \phi(a)+\phi(b) = \phi(c)+\phi(d) \end{cases}$$ □

LEMMA 7.9 $\exists \eta, \gamma$ dep only on α, and AP. P of length $\geq N^\gamma$ s.t.: with λ, μ.

$$\sum_{k\in P} |\hat{\Delta}(f;k)(\lambda k+\mu)|^2 \geq \eta N^2 |P|$$ $\gamma = \alpha^K$, $\eta = e^{-\alpha^K}$, K is const.

(P). Sketch, since Verstraëte's notes are missing details and precise statements anyway.

Recall that each $k\in B$ had actually $\|\hat{\Delta}(f;k)(\cdot)\|_\infty^2 \geq \frac{\alpha N^2}{2}$, so it suffices to show that:

If ϕ has $\geq \alpha N^3$ additive quadruples (ie $a+b=c+d$, $\phi(a)+\phi(b) = \phi(c)+\phi(d)$), $\phi: B \to \mathbb{Z}_N$ and $|B| = \beta N$

Then $\exists \gamma, \eta$ dep only on α, β, s.t. ϕ agrees with some linear function $\psi: B\to\mathbb{Z}_N$ on at least $\eta|P|$ points of some progression $P\subseteq B$, with $|P|\geq N^\gamma$.

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C

<u>Pf</u> of Statement.

Let Γ be graph of $\phi: B \to \mathbb{Z}_N$ but plotted in $\mathbb{Z}\times\mathbb{Z}$.

So $|\Gamma| = \beta N$.

We assumed that Γ has high additive energy: $\geq \alpha N^3$.

So by Balog-Szemeredi-Gowers, $\exists\, \Gamma' \subseteq \Gamma$ s.t. $|\Gamma'| \geq c|\Gamma|$

and $|\Gamma' + \Gamma'| \leq C|\Gamma'|$.

c and C depend only on α, and polynomially.

Note that $\Gamma' \subseteq \mathbb{Z}\times\mathbb{Z}$, so this is not exactly the Freiman thm we proved.

But by Freiman-type thm in $\mathbb{Z}^2$, we can say:

$$|\Gamma'+\Gamma'| \leq C|\Gamma'| \Rightarrow \exists\ d\text{-dimensional GAP of size} \leq C^{oC^{o(1)}}|\Gamma'|,\ Q.$$

such that $\Gamma' \subseteq Q$. $C^{o(1)}$

By renaming η, C, we conclude:

(*) $|Q \cap \Gamma| \geq \eta|Q| \leftarrow \eta = c\beta$.

$|Q| \leq CN \leftarrow C = C^{C^{o(1)}}\beta$.

Yet $Q = P_1 + \cdots + P_d$, each is 1-dim AP.

Since $Q \supseteq \Gamma'$ and $|\Gamma'| \geq c|\Gamma|$, $|Q| \geq \eta N$. since $|\Gamma| = \beta N$.

So, $\exists$ some $|P_i| \geq (\eta N)^{1/d}$.

"Foliate" Q into partition of disjoint translates of P_i.

(*) $\Rightarrow \exists$ translate R s.t. $|R \cap \Gamma| \geq \eta|R|$

<u>CHECK</u>: R not vertical: or else since Γ is graph of fcn, LHS $=1 \Rightarrow |R| \leq \eta^{-1} = O(1)$, ✗.

So let P = set of x-coords of R, we get a linear function supported on P,

with $|P| = |R| \geq (\eta N)^{1/d} \geq N^{\gamma}$,

and ϕ agrees with it on $|R \cap \Gamma| \geq \eta|R| = \eta|P|$ points. $\square$

2007-04-12

D

LEMMA 7.10 $f : \mathbb{Z}_N \to [-1,1]$, $\eta > 0$, $P \subseteq \mathbb{Z}_N$ is an AP s.t. for some $\lambda, \mu \in \mathbb{Z}_N$,

$$\sum_{k \in P} \left| \hat{\Delta}(f;k)\,(2\lambda k + \mu) \right|^2 \geq \eta |P| N^2, \text{ and } N^\gamma \leq |P| \leq N^{1/2}.$$

To avoid pathological cases

Then $\exists$ partition of $\mathbb{Z}_N$ into ~~translates~~ $P_1, P_2, \ldots, P_M$,

each of which is either translate of P or translate of P with endpt removed, s.t.

$$\sum_i \left| \sum_{x \in P_i} f(x)\, \omega^{-\lambda x^2 - r_i x} \right| \geq \eta \frac{N}{2}.$$

(For some choices of r_i)

$\omega = e^{\frac{-2\pi i}{N}}$, ~~apparently we must define~~

(P). $$\sum_{k \in P} \left| \hat{\Delta}(f;k)(2\lambda k + \mu) \right|^2 = \sum_{k \in P} \sum_{x,y} f(x)\overline{f(x-k)}\, \omega^{-(2\lambda k + \mu)(x)}\, \overline{f(y) \overline{f(y-k)} \omega^{-(2\lambda k + \mu) y}}$$

$$\leq \quad u = x - y \to \quad = \sum_{k \in P} \sum_{x,u} f(x)\overline{f(x-k)}\, \underset{y}{\overline{f(x-u)}}\, \underset{y}{f(x-u-k)}\, \omega^{-(2\lambda k + \mu) u}$$

$\eta |P| N^2$

Every $u \in \mathbb{Z}_N$ can be written in exactly $|P|$ ways as $v + l$ with $v \in \mathbb{Z}_N$, $l \in P$, so:

$$\eta |P|^2 N^2 \leq \sum_{k, l \in P} \sum_{x,v} f(x)\overline{f(x-k)}\, \overline{f(x-v-l)}\, f(x-v-l-k)\, \omega^{-(2\lambda k + \mu)(v + l)}$$

$\Rightarrow \exists v \in \mathbb{Z}_N$ s.t.

$$\eta |P|^2 N \leq \left| \sum_{k, l \in P} \sum_x f(x)\overline{f(x-k)}\, \overline{g(x-l)}\, g(x-k-l)\, \omega^{-(2\lambda k + \mu)(v + l)} \right|$$

$g(\cdot) := f(\cdot - v)$

$2\lambda k v + 2\lambda k l + \mu v + \mu l$.

discard since μ, v fixed, and we took abs. val.

Write: $2\lambda v k = 2\lambda v \left[(x-l) - (x-k-l) \right]$

$\mu l = \mu \left[(x) - (x-l) \right]$

$2\lambda k l = \lambda \left[(x)^2 - (x-k)^2 - (x-l)^2 + (x-k-l)^2 \right]$

$h_1(x) = f(x)\, \omega^{-\lambda x^2 - \mu x}$

$h_2(x) = f(x)\, \omega^{-\lambda x^2}$

$h_3(x) = g(x)\, \omega^{-\lambda x^2 + (2\lambda v - \mu)(x)}$

$h_4(x) = g(x)\, \omega^{-\lambda x^2 + 2\lambda v x}$

2007-04-12

(E)

Then the expression becomes precisely $\left|\sum_{k,l\in P}\sum_x h_1(x)\,\overline{h_2(x-k)}\,\overline{h_3(x-l)}\,h_4(x-k-l)\right|$.

$$\Rightarrow \sum_x \left|\sum_{k,l\in P} h_1(x)\,\overline{h_2(x-k)}\,\overline{h_3(x-l)}\,h_4(x-k-l)\right| \ge \eta|P|^2 N.$$

<u>DEF</u> $\eta(x):$

$$\underbrace{\left|\sum_{k,l\in P} h_1(x)\,\overline{h_2(x-k)}\,\overline{h_3(x-l)}\,h_4(x-k-l)\right|}_{\text{Use Circle Method: (and throw away } h_1(x) \text{ since } |h_1|\le 1.}=\eta(x)|P|^2.$$

$\sum \eta(x) \ge \eta N.$

$$\le \frac{1}{N}\left|\sum_r \sum_{k,l\in P}\sum_{m\in P+P} \overline{h_2(x-k)}\,\overline{h_3(x-l)}\,h_4(x-m)\,\omega^{r(k+l-m)}\right|$$

$$\le \frac{1}{N}\sum_r \left|\sum_{k\in P}\overline{h_2(x-k)}\,\omega^{-rk}\right|\cdot\left|\sum_{l\in P} h_3(x-l)\,\omega^{-rl}\right|\cdot\left|\sum_{m\in P+P} h_4(x-m)\,\omega^{-rm}\right|.$$

$$\le \frac{1}{N}\left(\max_r\left|\sum_{k\in P} h_2(x-k)\,\omega^{-rk}\right|\right)\left(\sum_r \left|\,—\,\right|\cdot\left|\,—\,\right|\right)$$

Yet $\sum_r \left|\sum_{l\in P} h_3(x-l)\,\omega^{-rl}\right|^2$

$$=\sum_r\sum_{l_1,l_2\in P} h_3(x-l_1)\,\overline{h_3(x-l_2)}\,\omega^{-r(l_1-l_2)}$$

(Circle Method) $= N\sum_{l\in P}\underbrace{|h_3(x-l)|^2}_{\le 1} \le N\cdot|P|$

and similarly for h_4, so by Cauchy-Schwarz:

$$\sum_r \left|\,—\,\right|\cdot\left|\,—\,\right| \le \sqrt{N|P|\times N\underbrace{|P+P|}_{\le 2|P|}}$$

$$\le \sqrt{2}\,N|P|$$

So:
$$\underbrace{\frac{\eta(x)|P|^2}{\frac{1}{N}\cdot\sqrt{2}N|P|}}_{=\frac{\eta(x)|P|}{\sqrt{2}}} \le \max_r\left|\sum_{k\in P} h_2(x-k)\,\omega^{-rk}\right|.$$

Let r_x be the max r.

So:
$$\frac{\eta(x)|P|}{\sqrt{2}} \le \left|\sum_{k\in P} f(x-k)\,\omega^{-\lambda(x-k)^2}\,\omega^{-r_x k}\right|$$

replace this with $\omega^{-r_x(x-k)}$ since $r_x\cdot x$ is const and $|\omega|=1$

Summing over all x, we get:

$$\eta|P|\cdot N\frac{1}{\sqrt{2}} \le \sum_x\left|\sum_{y\in x-P} f(y)\,\omega^{-\lambda y^2 - r_x y}\right|.$$

2007-04-12

F

Now if ($|P| \mid N$), then we could use a std. averaging argument to conclude:
← (but it doesn't since N prime)

(*) $\exists$ partition of $\mathbb{Z}_N$ into translates of P, called $P_1, \dots, P_M$, s.t.

$$\sum_i \left| \sum_{y \in P_i} f(y) \omega^{-\lambda y^2 - r_i y} \right| \geq \eta N \frac{1}{\sqrt{2}}.$$

But $|P| \nmid N$. So we partition $\mathbb{Z}_N$ into either translates of $\left\{ \begin{array}{l} P \\ \text{or} \\ P \text{ with endpt removed} \end{array} \right.$

Note that if we know something about $\sum_x \left| \sum_{y \in P} f(y) \omega^{-\lambda y^2 - r_x y} \right| \geq \eta N |P| \frac{1}{\sqrt{2}}$,

then if we delete an endpoint, this will effect the value by at most N, since $\|f\| \leq 1$.

Proportionally it will effect by at most $\frac{N}{\eta |P| \frac{1}{\sqrt{2}} N} = \frac{\sqrt{2}}{\eta |P|}$.

But η is const and $|P| \geq N^{-\gamma} \to \infty$.

So we may make the desired conclusion by reducing $\frac{1}{\sqrt{2}}$ to $\frac{1}{2}$:

(**) $\exists$ partition of $\mathbb{Z}_N$ into translates of $\left\{ \begin{array}{l} P \\ \text{or} \\ P \text{ with endpt removed} \end{array} \right.$, called $P_1, \dots, P_M$,

$$\sum_i \left| \sum_{y \in P_i} f(y) \omega^{-\lambda y^2 - r_i y} \right| \geq \eta N \frac{1}{2}.$$

(Don't give full details because that is not the important point of the proof)

2007-04-16 (A)

Now we know that $f(x)$ is correlated with a quadratic phase. Want to show:

~~LEMMA 7.13 Let $\phi(x)$ be quadratic poly and $r \le N$. THEN $\exists\, m \le O(r^{1-\frac{1}{8 \times 128}})$ s.t. $[0, r-1]$ can be partitioned~~

If $\left|\sum_{x \in Q} f(x)\, \omega^{\phi(x)}\right| \geq \nu |Q|$

$\phi(x)$: quadratic poly

Q: $\mathbb{Z}_N$-AP, of length $\gtrsim N^{\gamma}$.

THEN $\exists$ partition of Q into $\mathbb{Z}$-AP's $Q_1, \ldots, Q_m$ s.t. we control their length (from above) and

$$\sum_{i=1}^{m} \left| \sum_{x \in Q_i} f(x) \right| \geq \frac{\nu |Q|}{2}.$$

Q_i: $\mathbb{Z}$-AP

(Since $f = A - (\text{density of } A)$, this will lead to increased density on a $\mathbb{Z}$-subprogression)

Def. $\phi : \mathbb{Z}_N \to \mathbb{Z}_N$, $S \subseteq \mathbb{Z}_N$. $\quad \operatorname{diam} \phi(S) = \max_{x,y \in S} |\phi(x) - \phi(y)|$ ← dist. taken modulo N.

e.g. if in $\mathbb{Z}_5$, dist between 4 and 0 is 1.

Lem 7.11 $m, r, l \in [N]$, P is $\mathbb{Z}_N$-AP of length m. $\phi : \mathbb{Z}_N \to \mathbb{Z}_N$ is linear function. Say $l \le \left(\frac{m}{r}\right)^{1/3}$. $\Rightarrow$ P can be partitioned into $\mathbb{Z}_N$-subprogressions P_i of lengths l or $l-1$, s.t. $\operatorname{diam} \phi(P_i) \le \frac{N}{r}$ all.

(Pf) WLOG, $P = [0, m-1]$.

Pigeonhole $\Rightarrow \exists\, d \le rl$ s.t. $|\phi(d) - \phi(0)| \le \frac{N}{rl}$ ← dist. in $\mathbb{Z}_N$.

Let $Q = \{0, d, \ldots, (l-1)d\}$ ← length l AP that has $\operatorname{diam} \phi(Q) \le l \times \frac{N}{rl} = \frac{N}{r}$, as req'd.

To perform partition of P, note that any translate of Q has the $\operatorname{diam} \phi(Q)$ property since ϕ linear.

So: ① Cut $[0, m-1]$ into congruence classes mod d.

Each class has size $\geq \left\lfloor \frac{m}{d} \right\rfloor \gtrsim \frac{m}{rl} \geq l^2$ (assumption), so:

② In each congruence class we may partition the $\geq l^2$ elements into copies of either Q or (Q-endpt.)

(l^2 is because every l units gives us room to choose $\langle {l \atop l-1}$, so after l^2 we can get any residue)

<u>Lem 7.12</u> $\phi: \mathbb{Z}_N \to \mathbb{Z}_N$ quadratic poly.
P: $\mathbb{Z}_N$-AP of length m
$l \le m^{\frac{1}{18\times128}}$
$\Rightarrow$ P can be partitioned into $\mathbb{Z}_N$-subprogressions of lengths l or $l-1$, and all diam $\phi(P_i) \le \cancel{O(m^{-\frac{1}{6\times128}} N)}$.
$\le 3 m^{-\frac{1}{6\times128}} N$.

(P). WLOG, $P = [m]$
Let $\phi(x) = ax^2 + bx + c$.

distance to nearest integer.

[<u>Recall WEYL</u>: $\forall k, \exists \varepsilon > 0$ s.t. $\forall$ suff. large M and $\alpha \in \mathbb{R}$, $\exists q \le M$ s.t. $\|q^k \alpha\| \le 2M^{-\varepsilon}$.
For $k=2$, can take $\varepsilon = \frac{1}{64}$.

So $\exists d \le \sqrt{m}$ s.t. $\|d^2 \frac{a}{N}\| \le 2 (\sqrt{m})^{-\frac{1}{64}}$

$\Rightarrow \|d^2 a\| \le 2 m^{-\frac{1}{128}} N$.

Now split into congruence classes mod d, and consider
$Q = \{0, d, \ldots, (t-1)d\}$ with $t \approx m^{\frac{1}{3\times128}}$. (either take t as this value, or as the value -1)
We may partition each congruence class into translates of $\left\langle \begin{array}{l} Q \\ Q\text{-endpoint} \end{array}\right.$, since:
Size of any class $\ge \lfloor \frac{m}{d} \rfloor \gtrsim \sqrt{m}$,
but $|Q| = t \approx m^{\frac{1}{3\times128}}$. ok.

Now we have P = union of translates of Q and Q-endpts
But within any $x+Q$, diam ϕ is bdd by:

$$|\phi(x+td) - \phi(x)|_{\text{mod } N} \le |a(td)^2| + |t(2axd + bd)|$$
$$\le \underbrace{t^2 \cdot 2m^{-\frac{1}{128}} N}_{\text{error.}} + \underbrace{t(2axd + bd)}_{\text{linear function in } t, \text{ call it } \psi.}$$

Since we chose $t \approx m^{\frac{1}{3\times128}}$, error $\le 2 m^{-\frac{1}{3\times128}} N$.

So consider some fixed $x+Q$, and use <u>Lem 7.11</u> with ψ to partition it into $\mathbb{Z}_N$-subprogressions of length l or $l-1$, s.t. diam $\psi \le m^{-\frac{1}{6\times128}} N$. Check conditions:
~~We assumed that~~ We need $l \le \left(\frac{\text{length of } x+Q}{m^{\frac{1}{6\times128}}}\right)^3 \approx \left(m^{\frac{1}{6\times128}}\right)^3$, which is exactly what we assumed

So we can partition further, and total diam $\phi \le 2m^{-\frac{1}{3\times128}} N + m^{-\frac{1}{6\times128}} N \cancel{= O(m^{-\frac{1}{6\times128}} N)}$
$\le 3 m^{-\frac{1}{6\times128}} N$. $\square$

2007-04-16

C

So from LEM 7.12, we cut P into $\mathbb{Z}_N$-AP's of length l or $l-1$, $l \approx m^{\frac{1}{18\times 128}}$,
apply with this l.

such that each has diam $\phi \le$ ~~...~~

$\le 3\, m^{-\frac{1}{6\times 128}} N$.

We also began with $|P| \gtrsim N^{\gamma}$.

Now LEM 2.7 from Roth:

~~...~~, $\mathbb{Z}_N$-AP: $\{a, a+d, \dots, a+(m-1)d\}$ of length m can be partitioned into $\le 3\sqrt{m}$ $\mathbb{Z}$-AP's.

(recall pf: let $l \approx \sqrt{m}$, and use pigeonhole to find $s \le l$ st $|sd|_{\mathrm{mod}\, N} \le \frac{N}{l}$.
Then split the $\{a, \dots, a+(m-1)d\}$ into s classes, each is $\mathbb{Z}_N$-AP of common diff. sd, so it has $\mathbb{Z}$-runs of length $\ge \frac{N}{sd} \ge l$,
and up to 2 residual runs (starting & finishing).

~~Collect long runs~~)

So we may further subpartition the $\mathbb{Z}_N$-AP's, and get a family $P = P_1 \cup \dots \cup P_M$ of $\mathbb{Z}$-AP's, of average length $\approx \dfrac{l}{3\sqrt{l}} \approx \dfrac{1}{3} m^{\frac{1}{2\times 18\times 128}}$, and still diam $\phi(P_i) \le 3 m^{-\frac{1}{6\times 128}} N$.

Don't forget $m = |P| \ge N^{\gamma}$.

LEMMA 7.13 ~~$f: \mathbb{Z}_N \to \mathbb{Z}_N$ quadratic poly~~ ~~$r \gtrsim N^{\gamma}$~~ Subject to above, if $\left|\sum_{x\in P} f(x)\, \omega^{-\phi(x)}\right| \ge \eta |P|$,
then this partition satisfies $\sum_i \left|\sum_{x \in P_i} f(x)\right| \ge \frac{\eta}{2}|P|$.

(P) Note diam $\phi(P_i) \le 3\left(N^{\gamma}\right)^{-\frac{1}{6\times 128}} N \ll N$, so for suff. large N, this is $\le \frac{\eta N}{4\pi}$.

$$\Rightarrow \underset{x,y\in P_i}{\left|\omega^{\phi(x)} - \omega^{\phi(y)}\right|} \le \left|e^{\frac{2\pi i}{N}\frac{\eta N}{4\pi}} - 1\right| \le \frac{\eta}{2}.$$

So:

$$\eta |P| \le \left|\sum_P f(x)\, \omega^{-\phi(x)}\right| \le \sum_i \left|\sum_{P_i} f(x)\, \omega^{-\phi(x)}\right|$$

$$\le \sum_i \left(\left|\sum_{P_i} f(x)\right| + \underbrace{|P_i| \frac{\eta}{2}}_{\text{since } |f| \le 1} \right)$$

since we may choose one rep $\omega^{-\phi(x_i)}$ and pull it out.

$$\Rightarrow \sum_i \left|\sum_{P_i} f(x)\right| \ge \frac{\eta}{2} |P| \quad \text{since } \sum_i |P_i| = |P|.$$

FINISH

2007-04-16

(1)

$A \subseteq [N]$

SZEMERÉDI 4-AP. $\exists c > 0$ s.t. if $|A| = \delta N$ and $\delta \geq (\log\log\log N)^c$, then A has 4-AP.

(Pf). We were done if A was quadratically $\alpha = \frac{\delta^{32}}{2^{88}}$ - uniform.

Let $f(x) = A(x) - \delta$, and suppose NOT quad α-unif.

Lem 7.7 $\Rightarrow \exists B \subseteq \mathbb{Z}_N$, size $|B| \geq \frac{\alpha N}{2}$, and fcn $\phi: B \to \mathbb{Z}_N$ s.t.

$$\sum_{k \in B} |\hat{\Delta}(f;k)(\phi(k))|^2 \geq \left(\frac{\alpha}{2}\right)^2 N^3$$

Lem 7.8 $\Rightarrow$ ϕ has $\geq \left(\frac{\alpha}{2}\right)^8 N^3$ additive quadruples

Lem 7.9 $\Rightarrow \exists$ $\mathbb{Z}_N$-AP, P, of length $|P| \geq N^{\gamma}$ s.t.

$$\sum_{k \in P} |\hat{\Delta}(f;k)(2\lambda k + \mu)|^2 \geq \eta |P| N^2$$

with $\gamma = \alpha^{O(1)} = \delta^{O(1)}$ and $\eta = \exp(-\alpha^{O(1)}) = \exp(-\delta^{O(1)})$.

Lem 7.10 $\Rightarrow$ Partition $\mathbb{Z}_N$ into translates of P or P-endpts s.t.

$$\sum_i \left| \sum_{x \in P_i} f(x)\, \omega^{-\lambda x^2 - r_i x} \right| \geq \frac{\eta N}{2}.$$

Lem 7.13 $\Rightarrow$ Further partition s.t. we have $\mathbb{Z}_N = Q_1 \cup \dots \cup Q_M$, and avg. length of Q_i is $\approx \frac{1}{3} m^{\frac{1}{2 \times 18 \times 128}}$ $\geq \frac{1}{3} N^{\frac{\gamma}{2 \times 18 \times 128}}$

and $$\sum_i \left| \sum_{x \in Q_i} f(x) \right| \geq \frac{\eta N}{4}$$

$$= \sum_i \left(\left|\sum_{Q_i} f(x)\right| + \sum_{Q_i} f(x) \right)$$ since $\sum f(x) = 0$.

$\Rightarrow$ # of Q_i's is $M \leq 3N^{1 - \frac{\gamma}{2 \times 18 \times 128}}$.

Now we are almost done. We get increased density, and just must make sure that the Q_i is ~~large~~ long enough

Total contribution from $|Q_i| \leq N^{\frac{\gamma}{4 \times 18 \times 128}}$ is $\leq 2(M) \cdot N^{\frac{\gamma}{4 \times 18 \times 128}} \ll N$ still.
↑ # Q_i total.

So, $$\sum_{i : |Q_i| > N^{\frac{\gamma}{4 \times 18 \times 128}}} \left(\underbrace{\left| \sum_{x \in Q_i} f(x) \right|}_{\text{call } \alpha_i} + \underbrace{\sum_{x \in Q_i} f(x)}_{\alpha_i} \right) \geq \frac{\eta N}{8}.$$

Familiar argument to show some $\alpha_i \geq \cdots$

$$\sum_{\text{good } i} (|\alpha_i| + \alpha_i) \geq \frac{\eta N}{8} \geq \frac{\eta}{8} \sum_{\text{good } i} |Q_i|.$$

$\Rightarrow \exists$ good i s.t. $(|\alpha_i| + \alpha_i) \geq \frac{\eta}{8}|Q_i| \Rightarrow \sum_{x \in Q_i} f(x) \geq \frac{\eta}{16}|Q_i|$. $\rightarrow$ density increases

$$\boxed{\begin{array}{l} \delta \to \delta + \frac{\eta}{16} \\ = \delta + \exp(-\delta^{O(1)}). \end{array}}$$

$$\boxed{\begin{array}{l} \text{on subprogression of length} \\ \geq N^{\frac{\gamma}{4 \times 18 \times 128}} = N^{\delta^{O(1)}} \end{array}}$$

2007-04-16

(E)

Calculate dependency on δ and N:

Density: $\delta \to \delta + \cancel{\exp(-\delta^c)}\cdot \frac{1}{\delta^{\delta^c}}$ $\quad$ c is some integer $c > 1$.

Length: $N \to N^{\delta^c}$ $\quad$ Can iterate as long as N is $\geq C$ ← some constant.

since we start with $\delta = \frac{1}{(\log\log\log N)^c}$.

How many steps to reach density 1?

$\delta \mathrel{*}= (1 + \exp(-\delta^c))$.

So in $\frac{1}{\exp(-\delta^c)}$ steps, double the density.

$$\frac{dy}{dx} = e^{-y^c} \qquad y<1 \qquad 0 < y^c < 1.$$

$$\frac{dy}{e^{-y^c}} = dx. \qquad d(e^{y^c}) = e^{y^c} c y^{c-1}\, dy$$

$$dy\, e^{y^c} = dx$$

$$\approx e^{y^c} = x$$

$$\approx y^c \approx \log x.$$

$$N^{\frac{c}{\log^c}} \sim e^{-c}$$

$$N^{(\delta^c)^{\frac{1}{\exp(-\delta^c)}}} = N^{(\delta^c)^{\exp(\delta^c)}} = N^{\delta^{\delta^c}} \quad \delta > (\log\log\log N)^{-c}.$$

$$\approx N^{e^{e^{\delta^c}}}$$

$$> N^{e^{e^{(\log\log\log N)^{-1}}}}$$

$$= e^{e^{e^{(\log\log\log N)^{-1}(\log\log\log N)}}}$$