

Lower Semicontinuity in the Calculus of Variations

Qualifying Oral Examination

Pan Liu

Carnegie Mellon University

panl@andrew.cmu.edu

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Main Question:

If $u_n \rightarrow u$ with respect to a “Sobolev-type” topology, then

$$\liminf_{n \rightarrow \infty} \int_{\Omega} f(x, u_n, \nabla u_n) dx \stackrel{???}{\geq} \int_{\Omega} f(x, u, \nabla u) dx.$$



Overview

It is well known that

$$f(x, u, \cdot) \text{ is quasiconvex} \Leftrightarrow u \mapsto \int_{\Omega} f(x, u, \nabla u) dx \text{ is s.w.l.s.c,}$$

where

- ▶ $\Omega \subset \mathbb{R}^N$ open, bounded;
- ▶ $u_n \rightharpoonup u$ with respect to a “Sobolev-type” topology;
- ▶ f satisfies some regularity (e.g. Carathéodory) and growth conditions.



Background

There is an extensive body of literature in this field, for example

- ▶ Scalar-Valued Case: Buttazzo, Ekeland, Struwe, etc.
- ▶ Vector-Valued Case: Morrey, Dacorogna, Ball, etc.



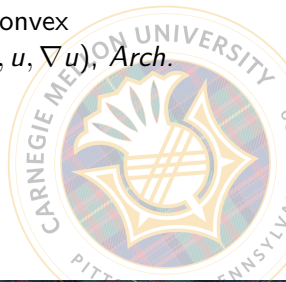
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Here we concentrate on the following papers:

- ▶ Fonseca, I., Müller, S. - Relaxation of Quasiconvex Functionals in $BV(\Omega, \mathbb{R}^p)$ for Integrands $f(x, u, \nabla u)$, *Arch. Rational Mech. Anal.* **123** (1993), 1 – 49.



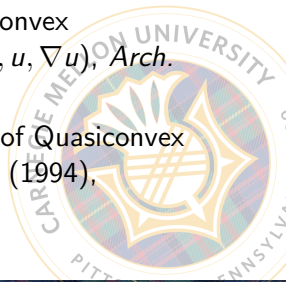
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- ▶ Ambrosio, L. - On the Lower Semicontinuity of Quasiconvex Integrals in $SBV(\Omega, \mathbb{R}^k)$, *Nonlinear Anal.* **23** (1994), 405 – 425.



Outline

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2. Proof of the s.w.l.s.c. of $\int_Q f(\nabla u) dx$ in the case $u_n \rightharpoonup \xi x$ in $W^{1,1}(Q; \mathbb{R}^d)$.



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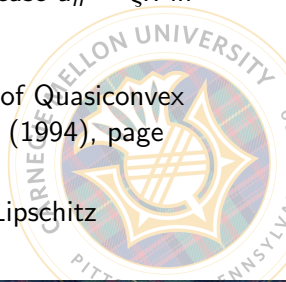
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Statement of the main theorem. Proof of a Lipschitz extension theorem for BV functions.



Definition of Quasiconvex Function

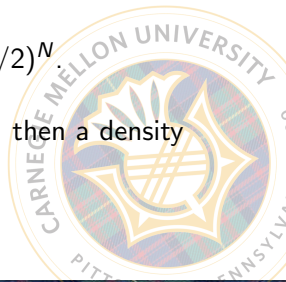
Definition:

For $N, d > 1$, a Borel measurable function $f: \mathbb{R}^{d \times N} \rightarrow \mathbb{R}$ is said to be *quasiconvex* if for all $\xi \in \mathbb{R}^{d \times N}$

$$f(\xi) \leq \frac{1}{|Q|} \int_Q f(\xi + \nabla \phi(x)) dx \quad (1)$$

for every $\phi \in W_0^{1,\infty}(Q; \mathbb{R}^d)$, where $Q = (-1/2, 1/2)^N$.

Remark: If $0 \leq f(\xi) \leq C(1 + |\xi|^p)$, $p \in [1, \infty)$, then a density result allows for ϕ in (1) to be $\phi \in W_{\text{per}}^{1,p}(Q; \mathbb{R}^d)$.



Objective

Objective: To obtain an integral representation in $BV(\Omega; \mathbb{R}^d)$ for the relaxed energy $\mathcal{F}(\cdot)$ of

$$u \in BV(\Omega; \mathbb{R}^d) \mapsto \int_{\Omega} f(x, u, \nabla u) dx,$$

$$\mathcal{F}(u) := \inf_{\{u_n\}} \left\{ \liminf_{n \rightarrow +\infty} \int_{\Omega} f(x, u_n(x), \nabla u_n(x)) dx \mid \{u_n\} \subset W^{1,1}(\Omega) \right. \\ \left. \text{and } u_n \rightarrow u \text{ in } L^1_{\text{loc}} \right\},$$

i.e., to identify the relaxed energy density \bar{f} s.t.

$$\mathcal{F}(u) = \int_{\Omega} \bar{f}(x, u, \nabla u) dx.$$



Main Result

The integral representation of relaxed energy $\mathcal{F}(u)$ is given by,

$$\begin{aligned}\mathcal{F}(u) = & \int_{\Omega} f(x, u, \nabla u) dx \\ & + \int_{\Sigma(u)} K(x, u^-, u^+, \nu) d\mathcal{H}^{N-1} \\ & + \int_{\Omega} f^{\infty} \left(x, u, \frac{dC(u)}{d|C(u)|} \right) d|C(u)|.\end{aligned}$$

Remark: The case $u \rightarrow \int_{\Omega} f(\nabla u) dx$ has been studied by Ambrosio & Dal Maso [1992].



Hypotheses on $f : \Omega \times \mathbb{R}^d \times \mathcal{M}^{d \times N} \rightarrow [0, +\infty)$

- ▶ $f(x, u, \cdot)$ is quasiconvex.



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- ▶ $f(x, u, \cdot)$ is quasiconvex.
- ▶ f has linear growth, i.e., $\exists C > 0$ s.t.

$$C^{-1} \|\xi\| \leq f(x, u, \xi) \leq C(1 + \|\xi\|);$$

For example:

$$f(\xi) = \sqrt{1 + |\xi|^2}.$$



Hypotheses on $f : \Omega \times \mathbb{R}^d \times \mathcal{M}^{d \times N} \rightarrow [0, +\infty)$

- ▶ $\forall K \in \Omega \times \mathbb{R}^d, \exists \omega \in C^0(\mathbb{R}), \omega(0) = 0$ such that if $(x, u), (x', u') \in K$

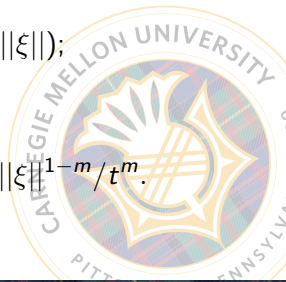
$$|f(x, u, \xi) - f(x', u', \xi)| \leq \omega(|x - x'| + |u - u'|)(1 + \|\xi\|);$$

- ▶ $\forall x_0 \in \Omega, \forall \delta > 0, \exists \epsilon > 0$ s.t. if $|x - x_0| \leq \epsilon$, then

$$f(x, u, \xi) - f(x_0, u, \xi) \geq -\delta(1 + \|\xi\|);$$

- ▶ $\exists c', L > 0, 0 \leq m < 1$, s.t. for $t > L/\|\xi\|$

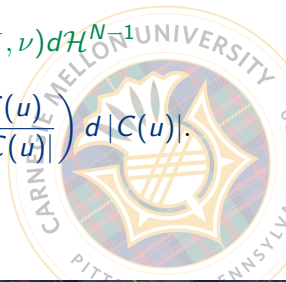
$$|f^\infty(x, u, \xi) - f(x, u, t\xi)/t| \leq c'g(x, u)\|\xi\|^{1-m}/t^m.$$



The Strategy of Proof - $u \in BV(\Omega)$

The integral representation of relaxed energy is a lower bound, i.e., for all sequence $\{u_n\}_{n=1}^{\infty} \subset W^{1,1}$, $u_n \rightarrow u$ in $L^1_{\text{loc}}(\Omega)$, with $u \in BV(\Omega; \mathbb{R}^d)$, then

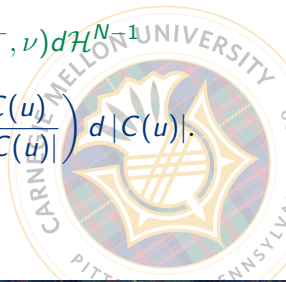
$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\Omega} f(x, u_n, \nabla u_n) dx &\geq \int_{\Omega} f(x, u, \nabla u) dx \\ &+ \int_{\Sigma(u)} K(x, u^-, u^+, \nu) d\mathcal{H}^{N-1} \\ &+ \int_{\Omega} f^{\infty} \left(x, u, \frac{dC(u)}{d|C(u)|} \right) d|C(u)|. \end{aligned}$$



The Strategy of Proof - $u \in BV(\Omega)$

The integral representation of relaxed energy is an upper bound, i.e., there exists a sequence $\{u_n\}_{n=1}^\infty \in W^{1,1}$, $u_n \rightarrow u$ in $L^1_{\text{loc}}(\Omega)$, with $u \in BV(\Omega; \mathbb{R}^d)$, then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\Omega} f(x, u_n, \nabla u_n) dx &\leq \int_{\Omega} f(x, u, \nabla u) dx \\ &+ \int_{\Sigma(u)} K(x, u^-, u^+, \nu) d\mathcal{H}^{N-1} \\ &+ \int_{\Omega} f^\infty \left(x, u, \frac{dC(u)}{d|C(u)|} \right) d|C(u)|. \end{aligned}$$



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Sketch of the proof of the lower bound using the blow-up method.
(The upper bound is skipped in this presentation.)



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Sketch of the proof of the lower bound using the blow-up method.
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Assume that

$$\liminf_{n \rightarrow +\infty} \int_{\Omega} f(x, u_n(x), \nabla u_n(x)) dx < \infty.$$

Up to a subsequence (not relabeled)

$$f(x, u_n(x), \nabla u_n(x)) \llcorner \Omega \xrightarrow{*} \mu$$

in the sense of measures, for some nonnegative finite Radon measure μ .



Lower Bound

Using the Radon-Nikodym Theorem we obtain

$$\mu = \mu_a \mathcal{L}_N + \xi |u^+ - u^-| \mathcal{H}^{N-1} \llcorner \Sigma(u) + \eta |C(u)| + \mu_s,$$

with $\mu_s \geq 0$. We will prove that

$$\mu_a(x_0) \geq f(x_0, u(x_0), \nabla u(x_0))$$

for a.e. $x_0 \in \Omega$;



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for $|u^+ - u^-| \mathcal{H}^{N-1} \llcorner \Sigma(u)$ a.e. $x_0 \in \Sigma(u)$.



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Density of the Absolutely Continuous Part

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Strategy:

Blow up argument:

- ▶ $\Omega \rightsquigarrow Q := (-1/2, 1/2)^N$;
- ▶ $u(x) \rightsquigarrow u_0(x) := u(x_0) + \nabla u(x_0)x$ (affine);



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Truncation and localization:

- ▶ $(x, u(x)) \rightsquigarrow (x_0, u(x_0))$;
- ▶ $f(x, u, \nabla u) \rightsquigarrow f_0(\nabla u) := f(x_0, u(x_0), \nabla u)$.



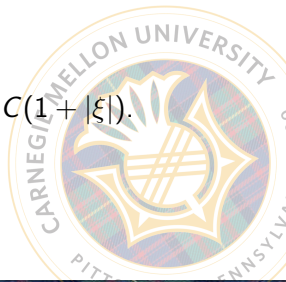
The Case $u_n \rightharpoonup \xi x$ in $W^{1,1}(Q)$

Theorem: [Bulk Case]

Let $\{u_n\}_{n=1}^\infty \subset W^{1,1}(Q; \mathbb{R}^d)$ be such that $u_n \rightharpoonup \xi x$ in $W^{1,1}(Q)$, where $Q = (-1/2, 1/2)^N$. Then

$$\liminf_{n \rightarrow \infty} \int_Q f(\nabla u_n) dx \geq f(\xi),$$

provided that $f(\cdot)$ is quasiconvex and $0 \leq f(\xi) \leq C(1 + |\xi|)$.

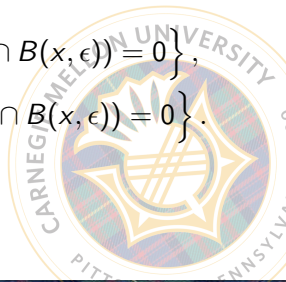


The Density of the Jump Part

Using the same strategy, for $|u^+ - u^-| \mathcal{H}^{N-1} \llcorner \Sigma(u)$ a.e. $x_0 \in \Sigma(u)$, we show that

$$\xi(x_0) \geq \frac{K(x_0, u^-(x_0), u^+(x_0), \nu(x_0))}{|u^+(x_0) - u^-(x_0)|}.$$

- ▶ $u_i^+ := \inf \left\{ t \in \mathbb{R} \mid \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon^N} \mathcal{L}^N(\{u_i > t\} \cap B(x, \epsilon)) = 0 \right\},$
- ▶ $u_i^- := \sup \left\{ t \in \mathbb{R} \mid \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon^N} \mathcal{L}^N(\{u_i < t\} \cap B(x, \epsilon)) = 0 \right\}.$



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- ▶ ν normal to $\Sigma(u) := \bigcup_{i=1}^d \{x \in \Omega \mid u_i^-(x) < u_i^+(x)\}$,
- ▶ ν exists \mathcal{H}^{N-1} -a.e. since $\Sigma(u)$ is rectifiable.

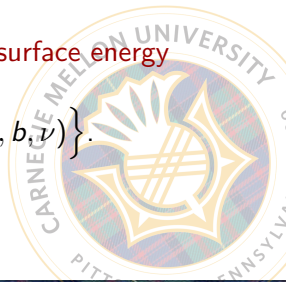


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- $K : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1} \rightarrow [0, +\infty)$ is the **surface energy density** $K(x, a, b, \nu) :=$
 $\inf_w \left\{ \int_{Q_\nu} f^\infty(x, w(y), \nabla w(y)) dy \mid w \in \mathcal{A}(a, b, \nu) \right\}.$

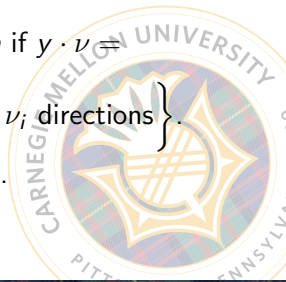


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- ▶ $\mathcal{A}(a, b, \nu) := \left\{ w \in W^{1,1}(Q_\nu; \mathbb{R}^d) \mid w(y) = a \text{ if } y \cdot \nu = -\frac{1}{2}, w(y) = b \text{ if } y \cdot \nu = \frac{1}{2}, w \text{ has period 1 in } \nu_i \text{ directions} \right\}.$
- ▶ Q_ν is a unit cube with two faces normal to ν .



The Density of the Cantor Part

Using the same strategy, for $|C(u)|$ a.e. $x_0 \in \Omega$, we show that

$$\eta(x_0) \geq f^\infty(x_0, u(x_0), A(x_0)).$$

► where

$$f^\infty(x, u, \xi) := \limsup_{t \rightarrow +\infty} \frac{f(x, u, t\xi)}{t}$$

is the recession function;

►

$$A(x_0) = \lim_{\varepsilon \rightarrow 0} \frac{Du(Q(x_0, \varepsilon))}{|D(u)| (Q(x_0, \varepsilon))}$$



Form Linear Growth to Superlinear Growth

- Linear growth: $c |\xi| \leq f(\xi) \leq C(1 + |\xi|)$

Start with a sequence $\{u_n\}_{n=1}^{\infty} \subset W^{1,1}(\Omega)$ and consider

$$\inf_{u_n \rightarrow u} \left\{ \liminf_{n \rightarrow \infty} \int_{\Omega} f(x, u_n, \nabla u_n), u_n \rightarrow u \text{ in } L^1_{\text{loc}} \right\},$$

for $u \in BV(\Omega)$.



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- ▶ Superlinear growth: $c |\xi|^p \leq f(\xi) \leq C(1 + |\xi|^p)$, some $p > 1$.



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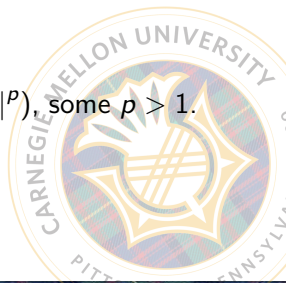
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- ▶ Superlinear growth: $c |\xi|^p \leq f(\xi) \leq C(1 + |\xi|^p)$, some $p > 1$.
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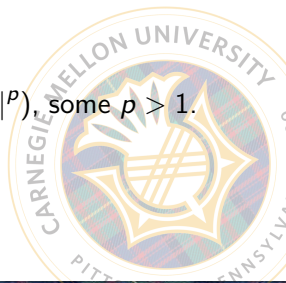
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for $u \in BV(\Omega)$.

- ▶ Superlinear growth: $c |\xi|^p \leq f(\xi) \leq C(1 + |\xi|^p)$, some $p > 1$.
 - ▶ What sequence should we start with?
 - ▶ What functional should we consider?



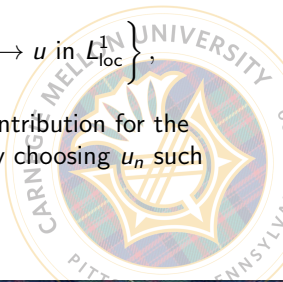
The Superlinear Case

If $c|\xi|^p \leq f(\xi) \leq C(1 + |\xi|^p)$, some $p > 1$, $u_n \rightarrow u$ in $L^1_{\text{loc}}(\Omega)$.

- ▶ If $u_n \in W^{1,p}(\Omega)$, then $u_n \rightarrow u$ in L^1_{loc} forces $u \in W^{1,p}(\Omega)$. Here l.s.c. and relaxation follow from Acerbi & Fusco.
- ▶ If $u_n, u \in BV$, $u_n \rightarrow u$ in $L^1_{\text{loc}}(\Omega)$, then
 - ▶ If we consider the relaxed energy

$$\inf_{u_n \in BV(\Omega)} \left\{ \liminf_{n \rightarrow \infty} \int_{\Omega} f(x, u_n, \nabla u_n) dx, u_n \rightarrow u \text{ in } L^1_{\text{loc}} \right\},$$

then it is impossible to obtain the energy contribution for the singular part of Du , e.g., if $f(x, u, 0) = 0$, by choosing u_n such that $\nabla u_n = 0$ a.e., we have nothing to do.



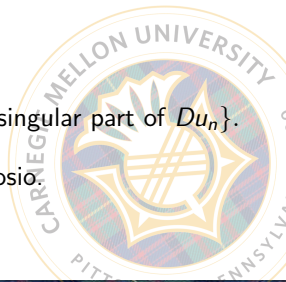
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 - ▶ If we consider the relaxed energy

$$\inf_{u_n \in BV(\Omega)} \left\{ \liminf_{n \rightarrow \infty} \int_{\Omega} f(x, u_n, \nabla u_n) dx \right. \\ \left. + \text{ term penalizing the singular part of } Du_n \right\}.$$

Here l.s.c. and relaxation follow from Ambrosio.



The SBV Case - Ambrosio

We consider the energy functional

$$\mathcal{E}(u) := \int_{\Omega} f(x, u, \nabla u) dx + \mathcal{H}^{N-1}(S(u)),$$

where $u_n \in SBV(\Omega)$ and $u_n \rightarrow u \in SBV(\Omega)$ in L^1_{loc} .

Suppose $\mathcal{E}(u_n)$ is bounded, we ask

$$\liminf_{n \rightarrow \infty} \int_{\Omega} f(x, u_n, \nabla u_n) dx \geq ???$$



The SBV Case - Ambrosio

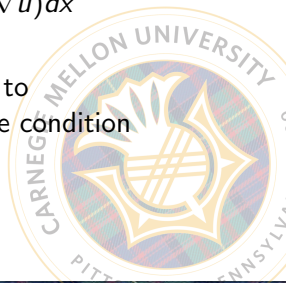
Theorem:

Let $f: \Omega \times \mathbb{R}^d \times M^{N \times d} \rightarrow [0, +\infty)$ be a Carathéodory function satisfying a super-linear growth condition, and assume that $f(x, s, \cdot)$ is quasiconvex in $M^{N \times d}$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^d$. Then we have

$$\liminf_{h \rightarrow \infty} \int_{\Omega} f(x, u_n, \nabla u_n) dx \geq \int_{\Omega} f(x, u, \nabla u) dx$$

for any sequence $\{u_n\} \subset SBV(\Omega, \mathbb{R}^d)$ converging to $u \in SBV(\Omega, \mathbb{R}^d)$ in $L^1_{loc}(\Omega, \mathbb{R}^d)$, and satisfying the condition

$$\sup_{n \in \mathbb{N}} \mathcal{H}^{N-1}(S_{u_n}) < +\infty.$$



Approximation of BV Functions.

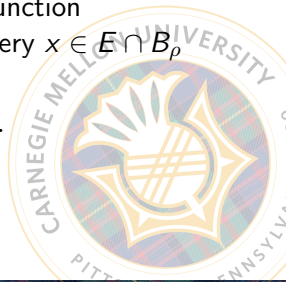
Theorem:

Let $\gamma > 0$, $B = B(0, 1)$, $u \in BV(B, \mathbb{R}^d) \cap L^\infty(B, \mathbb{R}^d)$, and

$$E := \{x \in B : M(|Du|)(x) \leq \gamma\}.$$

Then, for any $\rho \in (0, 1)$ we can find a Lipschitz function $v : B_\rho \rightarrow \mathbb{R}^d$ such that $u(x) = v(x)$ for almost every $x \in E \cap B_\rho$ and

$$\text{Lip}(v, B_\rho) \leq c(n)d\gamma + \frac{2d \|u\|_\infty}{1 - \rho}.$$



The Definition of Maximal Function.

Definition:

Let μ be a nonnegative, finite Radon measure in B . The maximal function $M(\mu)$ of μ is defined by

$$M(\mu)(x) := \sup \left\{ \frac{\mu(B_\rho(x))}{\mathcal{L}^N(B_\rho(x))} : 0 < \rho < 1 - |x| \right\}.$$

We have

$$\text{meas}(\{x \in B : M(\mu)(x) > \lambda\}) \leq \frac{c(n)\mu(B)}{\lambda}, \quad \forall \lambda > 0.$$



Thank you very much!



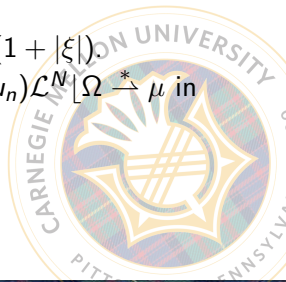
The Blow-up Method

Let $\{u_n\}_{n=1}^{\infty} \subset W^{1,1}(Q; \mathbb{R}^d)$ such that $u_n \rightarrow u$ in $W^{1,1}$, we consider the same question in Bulk case. i.e.,

$$\liminf_{n \rightarrow \infty} \int_{\Omega} f(\nabla u_n) dx \geq \int_{\Omega} f(\nabla u),$$

provided that f is quasiconvex and $0 \leq f(\xi) \leq C(1 + |\xi|)$.

Again, we consider the measure μ such that $f(\nabla u_n) \mathcal{L}^N \lfloor \Omega \xrightarrow{*} \mu$ in the sense of measures.



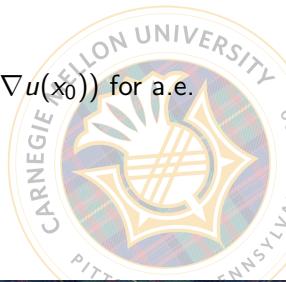
The Blow-up Method

By Radon-Nikodym, we have

$$\mu = \mu_a \mathcal{L}_N + \mu_s, \quad \mu_a(x_0) = \lim_{\varepsilon \rightarrow 0} \frac{\mu(Q(x_0, \varepsilon))}{\mathcal{L}_N(Q(x_0, \varepsilon))}$$

for a.e. $x_0 \in \Omega$.

We will be done once we proved that $\mu_a(x_0) \geq f(\nabla u(x_0))$ for a.e. $x_0 \in \Omega$. WLOG, we assume that $\nabla u(x_0) = 0$.



The Blow-up Method

We proceed to calculate, by choosing $\mu(\partial Q(x_0, \varepsilon_k)) = 0$,

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}_N}(x_0) &= \lim_{k \rightarrow \infty} \frac{\mu(Q(x_0, \varepsilon_k))}{\varepsilon_k^N} \\ &= \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\varepsilon_k^N} \int_{Q(x_0, \varepsilon_k)} f(\nabla u_n) dx \\ &= \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \int_Q f(\nabla v_{n,k}(y)) dy,\end{aligned}$$

where for $y \in Q$,

$$v_{n,k}(y) := \frac{u_n(x_0 + \varepsilon_k y)}{\varepsilon_k}.$$

