Lower Semicontinuity in the Calculus of Variations

Qualifying Oral Examination

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Main Question:

If $u_n \to u$ with respect to a "Sobolev-type" topology, then

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n)\, dx \overset{???}{\geq} \int_\Omega f(x, u, \nabla u)\, dx.$$

Overview

It is well known that

$$f(x,u,\cdot\,) \text{ is quasiconvex} \Leftrightarrow u \mapsto \int_\Omega f(x,u,\nabla u)dx \text{ is s.w.l.s.c,}$$

where

- $\Omega \subset \mathbb{R}^N$ open, bounded;
- $u_n \rightharpoonup u$ with respect to a "Sobolev-type" topology;
- f satisfies some regularity (e.g. Carathéodory) and growth conditions.

Background

There is an extensive body of literature in this field, for example

- Scalar-Valued Case: Buttazzo, Ekeland, Struwe, etc.
- Vector-Valued Case: Morrey, Dacorogna, Ball, etc.

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Here we concentrate on the following papers:

- Fonseca, I., Müller, S. - Relaxation of Quasiconvex Functionals in $BV(\Omega, \mathbb{R}^p)$ for Integrands $f(x, u, \nabla u)$, *Arch. Rational Mech. Anal.* **123** (1993), 1 – 49.

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- Ambrosio, L. - On the Lower Semicontinuity of Quasiconvex Integrals in $SBV(\Omega, \mathbb{R}^k)$, *Nonlinear Anal.* **23** (1994), 405 – 425.

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2. Proof of the s.w.l.s.c. of $\int_Q f(\nabla u)dx$ in the case $u_n \rightharpoonup \xi x$ in $W^{1,1}(Q; \mathbb{R}^d)$.

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3. Ambrosio, L. - On the Lower Semicontinuity of Quasiconvex Integrals in $SBV(\Omega, \mathbb{R}^k)$, *Nonlinear Anal.* **23** (1994), page $405 - 425$.

   Statement of the main theorem. Proof of a Lipschitz extension theorem for BV functions.

Definition of Quasiconvex Function

Definition:
For N, $d > 1$, a Borel measurable function $f\colon \mathbb{R}^{d\times N} \to \mathbb{R}$ is said to be *quasiconvex* if for all $\xi \in \mathbb{R}^{d\times N}$

$$f(\xi) \leq \frac{1}{|Q|}\int_Q f(\xi + \nabla\phi(x))dx \tag{1}$$

for every $\phi \in W_0^{1,\infty}(Q; \mathbb{R}^d)$, where $Q = (-1/2, 1/2)^N$.

Remark: If $0 \leq f(\xi) \leq C(1 + |\xi|^p)$, $p \in [1 + \infty)$, then a density result allows for ϕ in (1) to be $\phi \in W^{1,p}_{\text{per}}(Q;\, \mathbb{R}^d)$.

Objective

Objective: To obtain an integral representation in $\mathrm{BV}(\Omega;\mathbb{R}^d)$ for the relaxed energy $\mathcal{F}(\cdot)$ of

$$u \in \mathrm{BV}(\Omega;\mathbb{R}^d) \mapsto \int_\Omega f(x,u,\nabla u)dx,$$

$$\mathcal{F}(u) := \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))dx \mid \{u_n\} \subset W^{1,1}(\Omega) \right.$$
$$\left. \text{and } u_n \to u \text{ in } L^1_{\mathrm{loc}} \right\},$$

i.e., to identify the relaxed energy density $\bar{f}$ s.t.

$$\mathcal{F}(u) = \int_\Omega \bar{f}(x,u,\nabla u)dx.$$

Main Result

The integral representation of relaxed energy $\mathcal{F}(u)$ is given by,

$$\begin{aligned}\mathcal{F}(u) = &\int_\Omega f(x,u,\nabla u)dx \\ &+ \int_{\Sigma(u)} K(x,u^-,u^+,\nu)d\mathcal{H}^{N-1} \\ &+ \int_\Omega f^\infty\left(x,u,\frac{dC(u)}{d\left|C(u)\right|}\right)d\left|C(u)\right|.\end{aligned}$$

Remark: The case $u \to \int_\Omega f(\nabla u)dx$ has been studied by Ambrosio & Dal Maso [1992].

Hypotheses on $f : \Omega \times \mathbb{R}^d \times \mathcal{M}^{d \times N} \to [0, +\infty)$

- $f(x, u, \cdot)$ is quasiconvex.

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- $f(x, u, \cdot)$ is quasiconvex.
- f has linear growth, i.e., $\exists\ C > 0$ s.t.

$$C^{-1}||\xi|| \leq f(x, u, \xi) \leq C(1 + ||\xi||);$$

For example:

$$f(\xi) = \sqrt{1 + |\xi|^2}.$$

Hypotheses on $f : \Omega \times \mathbb{R}^d \times \mathcal{M}^{d\times N} \to [0, +\infty)$

- $\forall\ K \Subset \Omega \times \mathbb{R}^d$, $\exists\ \omega \in C^0(\mathbb{R})$, $\omega(0) = 0$ such that if $(x, u), (x', u') \in K$

$$|f(x, u, \xi) - f(x', u', \xi)| \le \omega(|x - x'| + |u - u'|)(1 + ||\xi||);$$

- $\forall\ x_0 \in \Omega$, $\forall \delta > 0$, $\exists \epsilon > 0$ s.t. if $|x - x_0| \le \epsilon$, then

$$f(x, u, \xi) - f(x_0, u, \xi) \ge -\delta(1 + ||\xi||);$$

- $\exists\ c', L > 0$, $0 \le m < 1$, s.t. for $t > L/||\xi||$

$$|f^\infty(x, u, \xi) - f(x, u, t\xi)/t| \le c' g(x, u)||\xi||^{1-m}/t^m.$$

The Strategy of Proof - $u \in BV(\Omega)$

The integral representation of relaxed energy is a lower bound, i.e., for all sequence $\{u_n\}_{n=1}^{\infty} \subset W^{1,1}$, $u_n \to u$ in $L^1_{\mathrm{loc}}(\Omega)$, with $u \in \mathrm{BV}(\Omega; \mathbb{R}^d)$, then

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n)\, dx \geq \int_\Omega f(x, u, \nabla u) dx + \int_{\Sigma(u)} K(x, u^-, u^+, \nu) d\mathcal{H}^{N-1} + \int_\Omega f^\infty \left(x, u, \frac{dC(u)}{d\,|C(u)|}\right) d\,|C(u)|.$$

The Strategy of Proof - $u \in BV(\Omega)$

The integral representation of relaxed energy is an upper bound, i.e., there exists a sequence $\{u_n\}_{n=1}^{\infty} \in W^{1,1}$, $u_n \to u$ in $L^1_{\text{loc}}(\Omega)$, with $u \in \text{BV}(\Omega;\mathbb{R}^d)$, then

$$\limsup_{n\to\infty} \int_\Omega f(x,u_n,\nabla u_n)\,dx \le \int_\Omega f(x,u,\nabla u)dx + \int_{\Sigma(u)} K(x,u^-,u^+,\nu)d\mathcal{H}^{N-1} + \int_\Omega f^\infty\left(x,u,\frac{dC(u)}{d\,|C(u)|}\right)d\,|C(u)|.$$

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Assume that

$$\liminf_{n \to +\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x)) dx < \infty.$$

Up to a subsequence (not relabeled)

$$f(x, u_n(x), \nabla u_n(x)) \lfloor \Omega \overset{*}{\rightharpoonup} \mu$$

in the sense of measures, for some nonnegative finite Radon measure μ.

Lower Bound

Using the Radon-Nikodym Theorem we obtain

$$\mu = \mu_a \mathcal{L}_N + \xi \left| u^+ - u^- \right| \mathcal{H}^{N-1} \lfloor \Sigma(u) + \eta \left| C(u) \right| + \mu_s,$$

with $\mu_s \geq 0$. We will prove that

$$\mu_a(x_0) \geq f(x_0, u(x_0), \nabla u(x_0))$$

for a.e. $x_0 \in \Omega$;

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$$\xi(x_0) \geq \frac{K(x_0, u^-(x_0), u^+(x_0), \nu(x_0))}{|u^+(x_0) - u^-(x_0)|}$$

for $|u^+ - u^-| \, \mathcal{H}^{N-1} \lfloor \Sigma(u)$ a.e. $x_0 \in \Sigma(u)$.

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$$\eta(x_0) \geq f^\infty(x_0, u(x_0), A(x_0))$$

for $|C(u)|$ a.e. $x_0 \in \Omega$.

Density of the Absolutely Continuous Part

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Strategy:

Blow up argument:

- $\Omega \rightsquigarrow Q := (-1/2, 1/2)^N$;
- $u(x) \rightsquigarrow u_0(x) := u(x_0) + \nabla u(x_0)x$ (affine);

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Truncation and localization:

- $(x, u(x)) \rightsquigarrow (x_0, u(x_0))$;
- $f(x, u, \nabla u) \rightsquigarrow f_0(\nabla u) := f(x_0, u(x_0), \nabla u)$.

The Case $u_n \rightharpoonup \xi x$ in $W^{1,1}(Q)$

Theorem: [Bulk Case]

Let $\{u_n\}_{n=1}^{\infty} \subset W^{1,1}(Q;\mathbb{R}^d)$ be such that $u_n \rightharpoonup \xi x$ in $W^{1,1}(Q)$, where $Q = (-1/2, 1/2)^N$. Then

$$\liminf_{n\to\infty} \int_Q f(\nabla u_n) dx \geq f(\xi)\,,$$

provided that $f(\cdot)$ is quasiconvex and $0 \leq f(\xi) \leq C(1+|\xi|)$.

The Density of the Jump Part

Using the same strategy, for $|u^+ - u^-|\,\mathcal{H}^{N-1}\lfloor\Sigma(u)$ a.e. $x_0 \in \Sigma(u)$, we show that

$$\xi(x_0) \geq \frac{K(x_0, u^-(x_0), u^+(x_0), \nu(x_0))}{|u^+(x_0) - u^-(x_0)|}.$$

- $u_i^+ := \inf\left\{ t \in \mathbb{R} \;\middle|\; \lim_{\epsilon\to 0^+} \frac{1}{\epsilon^N}\mathcal{L}^N\left(\{u_i > t\} \cap B(x,\epsilon)\right) = 0 \right\}$,
- $u_i^- := \sup\left\{ t \in \mathbb{R} \;\middle|\; \lim_{\epsilon\to 0^+} \frac{1}{\epsilon^N}\mathcal{L}^N\left(\{u_i < t\} \cap B(x,\epsilon)\right) = 0 \right\}$.

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- ν normal to $\Sigma(u) := \bigcup_{i=1}^{d}\{x \in \Omega \mid u_i^-(x) < u_i^+(x)\}$,
- ν exists $\mathcal{H}^{N-1}$–a.e. since $\Sigma(u)$ is rectifiable.

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- $K : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1} \to [0, +\infty)$ is the surface energy density $K(x, a, b, \nu) :=$ $\inf_w \left\{ \int_{Q_\nu} f^\infty(x, w(y), \nabla w(y))dy \;\middle|\; w \in \mathcal{A}(a, b, \nu) \right\}.$

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- $\mathcal{A}(a, b, \nu) := \Big\{ w \in W^{1,1}(Q_\nu; \mathbb{R}^d) \;\Big|\; w(y) = a \text{ if } y \cdot \nu = -\frac{1}{2}, w(y) = b \text{ if } y \cdot \nu = \frac{1}{2}, w \text{ has period 1 in } \nu_i \text{ directions} \Big\}$.
- Q_ν is a unit cube with two faces normal to ν.

The Density of the Cantor Part

Using the same strategy, for $|C(u)|$ a.e. $x_0 \in \Omega$, we show that

$$\eta(x_0) \geq f^\infty(x_0, u(x_0), A(x_0)).$$

- where
$$f^\infty(x, u, \xi) := \limsup_{t \to +\infty} \frac{f(x, u, t\xi)}{t}$$
is the recession function;
-
$$A(x_0) = \lim_{\varepsilon \to 0} \frac{Du(Q(x_0, \varepsilon))}{|D(u)|\,(Q(x_0, \varepsilon))}$$

Form Linear Growth to Superlinear Growth

- Linear growth: $c\,|\xi| \le f(\xi) \le C(1+|\xi|)$

  Start with a sequence $\{u_n\}_{n=1}^{\infty} \subset W^{1,1}(\Omega)$ and consider

$$\inf_{u_n \to u} \left\{ \liminf_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n), u_n \to u \text{ in } L^1_{\text{loc}} \right\},$$

  for $u \in BV(\Omega)$.

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- Superlinear growth: $c\,|\xi|^p \leq f(\xi) \leq C(1+|\xi|^p)$, some $p > 1$.
  - What sequence should we start with?
  - What functional should we consider?

The Superlinear Case

If $c\,|\xi|^p \leq f(\xi) \leq C(1+|\xi|^p)$, some $p>1$, $u_n \to u$ in $L^1_{\text{loc}}(\Omega)$.

- If $u_n \in W^{1,p}(\Omega)$, then $u_n \to u$ in L^1_{loc} forces $u \in W^{1,p}(\Omega)$. Here l.s.c. and relaxation follow from Acerbi & Fusco.
- If u_n, $u \in BV$, $u_n \to u$ in $L^1_{\text{loc}}(\Omega)$, then
  - If we consider the relaxed energy

$$\inf_{u_n \in BV(\Omega)} \left\{ \liminf_{n\to\infty} \int_\Omega f(x,u_n,\nabla u_n)dx, u_n \to u \text{ in } L^1_{\text{loc}} \right\},$$

then it is impossible to obtain the energy contribution for the singular part of Du, e.g., if $f(x,u,0)=0$, by choosing u_n such that $\nabla u_n = 0$ a.e., we have nothing to do.

The Superlinear Case

If $c\,|\xi|^p \le f(\xi) \le C(1+|\xi|^p)$, some $p>1$, $u_n \to u$ in $L^1_{\text{loc}}(\Omega)$.

- If $u_n \in W^{1,p}(\Omega)$, then $u_n \to u$ in L^1_{loc} forces $u \in W^{1,p}(\Omega)$. Here l.s.c. and relaxation follow from Acerbi & Fusco.
- If $u_n,\ u \in BV$, $u_n \to u$ in $L^1_{\text{loc}}(\Omega)$, then
  - If we consider the relaxed energy

$$\inf_{u_n \in BV(\Omega)} \left\{ \liminf_{n\to\infty} \int_\Omega f(x,u_n,\nabla u_n)dx \right.$$
$$\left. + \text{ term penalizing the singular part of } Du_n \right\}.$$

Here l.s.c. and relaxation follow from Ambrosio.

The SBV Case - Ambrosio

We consider the energy functional

$$\mathcal{E}(u) := \int_\Omega f(x, u, \nabla u)dx + \mathcal{H}^{N-1}(S(u)),$$

where $u_n \in SBV(\Omega)$ and $u_n \to u \in SBV(\Omega)$ in L^1_{loc}.

Suppose $\mathcal{E}(u_n)$ is bounded, we ask

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n)dx \geq ???$$

The SBV Case - Ambrosio

Theorem:

Let $f\colon \Omega \times \mathbb{R}^d \times M^{N\times d} \to [0,+\infty)$ be a Carathéodory function satisfying a super-linear growth condition, and assume that $f(x,s,\cdot)$ is quasiconvex in $M^{N\times d}$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^d$. Then we have

$$\liminf_{h\to\infty} \int_\Omega f(x,u_n,\nabla u_n)dx \geq \int_\Omega f(x,u,\nabla u)dx$$

for any sequence $\{u_n\} \subset SBV(\Omega,\mathbb{R}^d)$ converging to $u \in SBV(\Omega,\mathbb{R}^d)$ in $L^1_{loc}(\Omega,\mathbb{R}^d)$, and satisfying the condition

$$\sup_{n\in\mathbb{N}} \mathcal{H}^{N-1}(S_{u_n}) < +\infty.$$

Approximation of BV Functions.

Theorem:

Let $\gamma > 0$, $B = B(0,1)$, $u \in BV(B, \mathbb{R}^d) \cap L^\infty(B, \mathbb{R}^d)$, and

$$E := \{x \in B : \; M(|Du|)(x) \leq \gamma\} .$$

Then, for any $\rho \in (0,1)$ we can find a Lipschitz function $v : B_\rho \to \mathbb{R}^d$ such that $u(x) = v(x)$ for almost every $x \in E \cap B_\rho$ and

$$\mathrm{Lip}(v, B_\rho) \leq c(n) d\gamma + \frac{2d \, \|u\|_\infty}{1 - \rho} .$$

The Definition of Maximal Function.

Definition:

Let μ be a nonnegative, finite Radon measure in B. The maximal function $M(\mu)$ of μ is defined by

$$M(\mu)(x) := \sup\left\{ \frac{\mu(B_\rho(x)))}{\mathcal{L}^N(B_\rho(x))} : 0 < \rho < 1 - |x| \right\}.$$

We have

$$\text{meas}(\{x \in B : M(\mu)(x) > \lambda\}) \leq \frac{c(n)\mu(B)}{\lambda}, \ \forall \lambda > 0.$$

Thank you very much!

The Blow-up Method

Let $\{u_n\}_{n=1}^{\infty} \subset W^{1,1}(Q;\mathbb{R}^d)$ such that $u_n \to u$ in $W^{1,1}$, we consider the same question in Bulk case. i.e.,

$$\liminf_{n\to\infty} \int_\Omega f(\nabla u_n)dx \geq \int_\Omega f(\nabla u),$$

provided that f is quasiconvex and $0 \leq f(\xi) \leq C(1+|\xi|)$. Again, we consider the measure μ such that $f(\nabla u_n)\mathcal{L}^N\lfloor\Omega \overset{*}{\rightharpoonup} \mu$ in the sense of measures.

The Blow-up Method

By Radon-Nikodym, we have

$$\mu = \mu_a \mathcal{L}_N + \mu_s, \ \mu_a(x_0) = \lim_{\varepsilon \to 0} \frac{\mu(Q(x_0, \varepsilon))}{\mathcal{L}_N(Q(x_0, \varepsilon))}$$

for a.e. $x_0 \in \Omega$.

We will be done once we proved that $\mu_a(x_0) \geq f(\nabla u(x_0))$ for a.e. $x_0 \in \Omega$. WLOG, we assume that $\nabla u(x_0) = 0$.

The Blow-up Method

We proceed to calculate, by choosing $\mu(\partial Q(x_0, \varepsilon_k)) = 0$,

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}_N}(x_0) &= \lim_{k\to\infty} \frac{\mu(Q(x_0, \varepsilon_k))}{\varepsilon_k^N} \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{\varepsilon_k^N} \int_{Q(x_0,\varepsilon_k)} f(\nabla u_n) dx \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \int_Q f(\nabla v_{n,k}(y)) dy,
\end{aligned}$$

where for $y \in Q$,

$$v_{n,k}(y) := \frac{u_n(x_0 + \varepsilon_k y)}{\varepsilon_k}.$$