Sample Exam 1 Solutions

July 18, 2000

Question Two

(a) We'll let $P(n, r)$ denote the number of permutations r elements chosen from a total of n total elements. Thus $P(n, r) = n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!} = \binom{n}{r}r!$.

(i) $P(10, 4) = 10 \times 9 \times 8 \times 7 = 5040$

(ii) $1 \times 9 \times 8 \times 7 = 504$. We can also think of this as fixing the first element to 3, leaving us with the task of selecting 3 elements from 9 in which case the answer is $P(9, 3)$.

(iii) Same as above.

(b) (i) We'll leave the algebra as an exercise. Here's a combinatorial proof. The question asks us to prove

$$P(n, r) = P(n-1, r) + rP(n-1, r-1).$$

The left side counts all r-permutations of n elements. We'll break these into two groups, ones that cointain the "special" element 1 and those that do not. How many do not contain 1? We just pick our r-permutation from all the $n-1$ elements except 1, which gives us $P(n-1, r)$. How about those that do contain 1? In this case we'll pick an $(r-1)$-permutation from the $n-1$ elements except 1 and then insert 1 to give us an r-permutation. There are $P(n-1, r-1)$ such permutations, and we can insert 1 either before the first element, after the first, after the second, $\ldots$, or after the last, which gives us r ways to transform our $(r-1)$-permutation into an r-permutation containing 1. We've just demonstrated that

$$P(n, r) = P(n-1, r) + rP(n-1, r-1),$$

since every permutation must either contain 1 or not (and cannot do both).

(ii) Let the columns represent $r = 0, r = 1, \ldots, r = 6$. Use the convention that $P(n, 0) = 1$ along with the recurrence above and the fact that $P(n, n) = n!$.

r=	0	1	2	3	4	5	6
	1						
	1	1					
	1	2	2				
	1	3	6	6			
	1	4	12	24	24		
	1	5	20	60	120	120	
	1	6	30	120	360	720	720

(iii) $P(n, n-1) = \frac{n!}{1!} = \frac{n!}{0!} = P(n, n)$.

Question Three

(a) (i)-(iii) $J(1) = 1, J(2) = 2, J(3) = 3$

(iv) Each tiling must begin with either an Apricot or Burgundy tile but not both. There are $J(n-1)$ tilings that begin with an Apricot tile, and $J(n-2)$ tilings that begin with a Burgundy tile.

(b) (i)-(iii) $K(1) = 2, K(2) = 5, K(3) = 12$

(iv) This time we have $K(n-1)$ tilings when we begin with either a Canary or Dun tile, and $K(n-2)$ when we begin with an Ebony tile.

(c) (i)-(iii) $L(1) = 1, L(2) = 2, L(3) = 4$

(iv) We have $L(n-1)$ tilings that begin with a Forest tile, $L(n-2)$ that begin with a Gray tile, and $L(n-3)$ that begin with the other tile.

(d) (i) We have $M(n-i)$ tilings that start with the tile of length i, for $i = 1 \ldots n-1$, and one tiling with the tile of length n, for a total of

$$M(n) = 1 + \sum_{i=1}^{n-1} M(i) \text{ tilings.}$$

(ii) A base case is when $n = 1$ in which case we do indeed have $2^0 = 1$ tiling.

Now assume $M(i) = 2^{i-1}$ for $i \leq n$; we seek to show $M(n+1) = 2^n$. We have $M(n+1) = 1 + \sum_{i=1}^{n} M(i)$. By the induction hypothesis we can replace each $M(i)$ in the summand with 2^{i-1}, hence

$$M(n+1) = 1 + \sum_{i=1}^{n} 2^{i-1} = 1 + \sum_{j=0}^{n-1} 2^j, \text{ by setting } j = i-1.$$

Now we can apply the formula in the HINT to get

$$M(n+1) = 1 + (2^n - 1) = 2^n.$$

(iii) Let x_i represent the length of the ith tile from the left. We seek to find the number of solutions to

$$x_1 + x_2 + \ldots + x_k = n,$$

where $x_i \geq 1$ and is an integer for $1 \leq i \leq k$. This is a problem we've seen many times before. It is equivalent to distributing n indentical balls in k bins (let x_i represent the number of balls in bin i) where each bin must contain at least one ball. We can put one ball in each bin and then treat it like an unrestricted ball-bin problem to find there are

$$\binom{(n-k)+k-1}{n-1} = \binom{k-1}{n-1}$$

ways.

Question Four

(a) (i) We can choose the first group of two in $\binom{6}{2}$ ways and the second group of two in $\binom{4}{2}$ ways, but this counts each group of two twice [for instance Adam, Beth is counted by both terms], so we have $\frac{\binom{6}{2}\binom{4}{2}}{2}$ ways of choosing the two groups of two students. However after choosing these two groups of two we have two students left over who must each belong in her own group, so we have no more choices to make and are done.

(ii) There are two ways to let Zach and Yvonne supervise the two groups of two. There are also two choices for which group of one Xerxes supervises, so we have in total we have 4 ways to assign supervisors for each parition above, giving us

$$4\frac{\binom{6}{2}\binom{4}{2}}{2} = 2\binom{6}{2}\binom{4}{2}.$$

(b) There are $\binom{10}{3}$ ways of choosing a team of three and $\binom{7}{3}$ ways of choosing a team of three from the remainder, but as in part (i) above, each team is counted twice, so we must divide by 2. After choosing these teams we're left with four students who must all be in the same team, so we have no more choices to make. This gives us

$$\frac{\binom{10}{3}\binom{7}{3}}{2}$$

so far. Now for each team of three there are 3! ways of assigning the numbers, so our final answer is

$$\frac{\binom{10}{3}\binom{7}{3}}{2}3!3!.$$

(c) Let's pretend the pieces of chocolate are identical. In this case we have a balls-bins problem in which each bin must receive at least 2 balls. We can pre-fill each of the bins with 2 balls, and now we've reduced our problem to an unrestricted balls-bins problem with $10 - 2 \times 3 = 4$ balls, hence there are $\binom{4+(3-1)}{3-1}$ distributions. To account for the ordering of the chocolate we can simply count all the orders in which it could have been distributed, of which there are 10!, giving us a total of

$$\binom{4+(3-1)}{3-1}10!$$

distributions.

(d) (i) The value $f(x_i)$ can be $0, 1, \ldots$ for all i, and they all have to sum to k, so this is asking the number of solutions to

$$y_1 + y_2 + \ldots + y_n = k, \text{ where } y_i = f(x_i).$$

We know that we can solve this as a balls-bins problems (it strikes yet again) with k balls and n bins, so the answer is $\binom{n+k-1}{k}$.

(ii) We haven't talked about partitioning unlabeled objects into unlabed parts yet.

(iii) The scholarships are distinct and the parts (awards) are unnamed, so if we assume an award cannot be empty the answer is $S(k, n)$.

(iv) We'll assume the lottery tickets are identical. The contestants are not, and since contestants can receive 0 tickets, this is again a balls-bins'er: $\binom{n+k-1}{k}$.

(v) Let's count the number of values that can be assigned to each value in the domain. We have k choices for $f(1), f(2), \ldots, f(n)$, so in total we have k^n functions. We were able to use this approach since each domain element is mapped to exactly one range element.

(vi) Since the books are distinct, the shelves are not, and each part (shelf) must be non-empty, this is $S(k, n)$.

(vii) We have k distinct items, and the lots aren't labeled. Furthermore there is no specified number of lots, so this the number of ways we can divide k distinct elements into any number of undistinguished parts, which is $B_k = \sum_{i=0}^{k} S(k, i)$.