Department of Mathematics
Carnegie Mellon University

21-228 Discrete Mathematics, Summer II 2000: Exam 2 Solutions

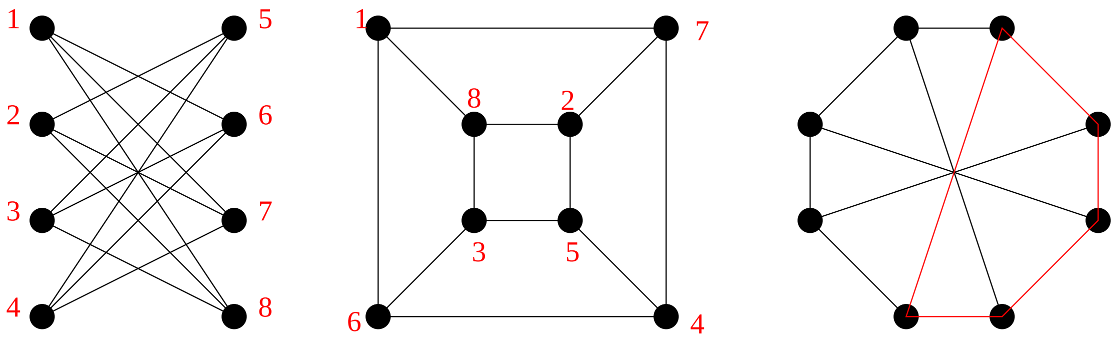
Name:

You have 90 minutes to complete this exam. Notes, books, and colloboration are prohibited.

Problem	Score	Total
Isomorphism & planarity		20
Trees & matchings		30
Everything Eulerian		40
Everything else		60
Bonus		15
		150

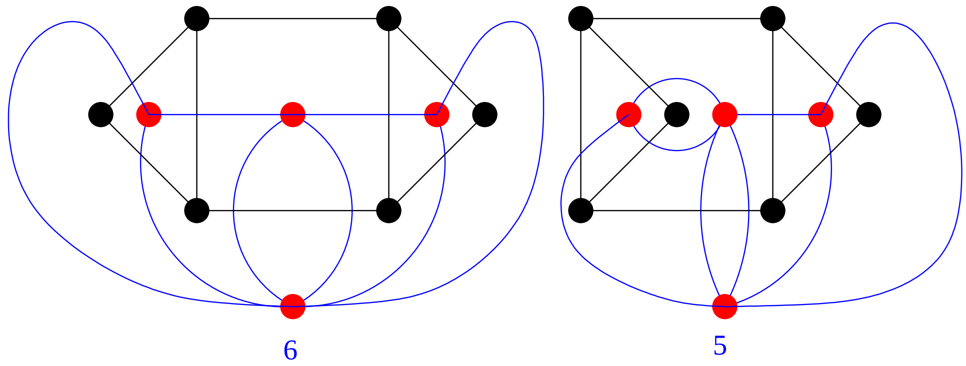
[20] Isomorphism and planarity

[10] Among the graphs below, which pairs are isomorphic? Justify your answer.



The two on the left are isomorphic as demonstrated by the labeling. Note that they are bipartite, but the one on the right contains and odd cycle.

[10] The plane graphs below are isomorphic. Show that their duals are not isomorphic.



The maximum degrees of the duals differ. This problem shows that the way a planar graph is drawn actually makes a difference. A planar graph G may have many different planar drawings, which we call *plane graphs*.

[30] Trees and matchings

A *spanning tree* of a graph $G = (V, E)$ is a subgraph that is a tree on the vertex set V. Recall that a *matching* of G is a subgraph of of maximum degree 1 (a collection of pairwise-disjoint edges). A matching M is *perfect* if for each vertex $u \in V$ there is some edge in M adjacent to u (every vertex is matched).

[**10**] Prove that the average degree of a vertex in a tree on n vertices is $2 - \frac{2}{n}$.

The average degree in G is $\frac{\sum_{v \in V} d(v)}{|V|} = \frac{2|E|}{|V|}$. For a tree we have $|E| = |V| - 1$, hence the average degree is $\frac{2(|V|-1)}{|V|} = 2 - \frac{2}{|V|}$.

[**10**] Describe each spanning tree of C_n, the cycle on n vertices. How many are there?

We know a spanning tree of G must have $|V| - 1$ edges. So every spanning tree of C_n must be obtained by removing an edge from C_n. The question is whether removing each of the n edges yields a disctinc spanning tree. If we remove the edge ij from C_n, we get a spanning path from i to j, hence removing each of the n edges yields a different spanning tree.

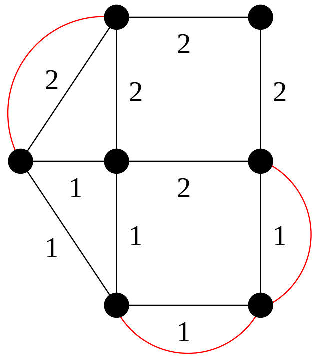
[**10**] Describe each perfect matching of C_n. How many are there?

If n is odd there are 0 perferct matchings, so suppose n is even. Since the degree of each vertex is 2, we only have two choices: either i gets matched with $i - 1$ or i gets matched with $i + 1$ (where we take $0 \equiv n$).

[40] Everything Eulerian

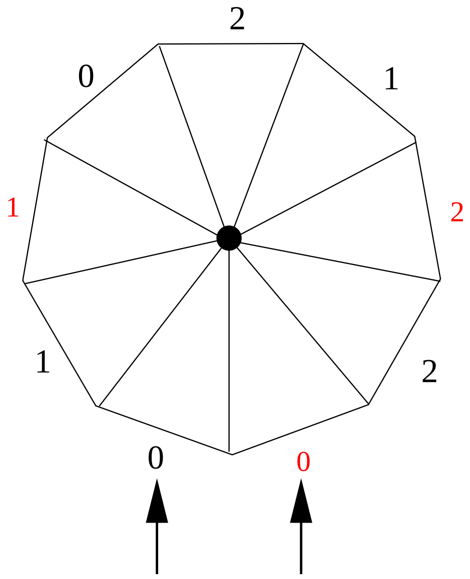
[10] Recall that given a connected graph $G = (V, E)$ and costs on the edges of E, the *Chinese Postman Problem* is that of finding a minimum-cost duplication of edges so that the resulting graph is Eulerian.

Solve the Chinese Postman Problem for the following graph.



We match the odd vertices as indicated by the red edges for an additional cost of 4.

[10] If the segments of the drum are labeled with elements from $\{0, 1, 2\}$ as shown below, can the remainder of the labels be filled in so that the position of the drum can be uniquely determined by reading the 2 indicated positions? Either produce a labeling, or briefly describe why the partial labeling below is inconsistent.



A consistent completion is given above.

[20] Prove that a connected graph G in which u and v are the only vertices of odd degree has a u,v-Eulerian walk, a walk from u to v that traverses each edge exactly once. Extend this to show that a connected graph G which has exactly $2k$ odd vertices, where $k \geq 1$, can be partioned into k walks, each between 2 odd vertices.

We will prove the extension by induction on $|E|$. Suppose G has n vertices and m edges. Our base case is $k = |E| = 1$, in which case the sole edge is a walk between the 2 odd vertices. Now suppose the claim holds for all graphs with $1 \leq k \leq \lfloor n/2 \rfloor$ and $|E| < m$.

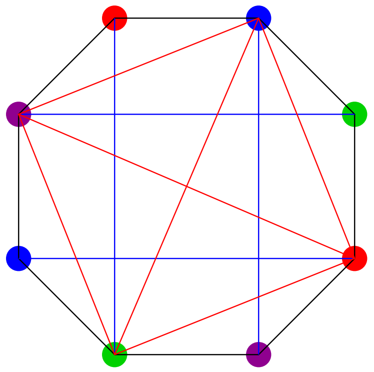
Pick some odd vertex $x \in X$ and delete some edge xu incident upon it; if xu is the only edge incident upon x, delete x as well. Let G' be the resulting graph. Deleting xu might disconnect the graph, but this is fine since we could apply the following argument to each component of G'.

We have two cases: either (i) $u \notin X$ or (ii) $u \in X$. If case (i) applies then u is odd in G', hence by the induction hypothesis we can partion G' into k walks joining the odd vertices, and we simply tack on xu to the walk containing u. If case (ii) applies then G' contains 2 fewer odd vertices than G. If x and u were the only odd vertices in G, then G' has none, but it does have an Eulerian circuit starting and ending at u. Thus when we add xu back in our x,u-Eulerian walk simply takes xu and then the Eulerian circuit at u. If G' has odd vertices, we apply the induction hypothesis to partion it into $k-1$ walks. Then xu simply serves as the kth walk in G.

Of course you need not worry about going into as much detail as above on future exams. Just make sure you understand and communicate the main idea of the proof.

[60] Everything else

A *vertex cover* in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that every edge in E is adjacent to some vertex in S. The set V is a trivial vertex cover. Recall that an *independent set* is a set of vertices $S \subseteq V$ such that $G[S]$, the subgraph of G induced by S, contains no edges.



[20] Find an independent set, matching, proper vertex coloring, and vertex cover in the graph O above, each of size 4.

The coloring is indicated above. The blue edges comprise the matching. The vertices of the red K_4 are a vertex cover, and the vertices not in the red K_4 are an independent set.

[10] Prove that your proper coloring of O is optimal.

It takes 4 colors just to color the vertices of the red K_4 above, so 4 is optimal.

[15] Let m be the size of a maximum matching and vc the size of a minimum vertex cover in some graph G. Show that $m \leq vc$. Use this fact to deduce that the matching and vertex cover you found in O are indeed optimal.

Consider the m edges of a matching. Any vertex cover must certainly cover these, but since these edges don't share any common vertices, it takes at least m vertices to cover them. Thus $m \leq vc$. Since we have $4 = m = vc = 4$, we know we have found an optimal matching and vertex cover. We also know our matching is optimal since it's perfect.

[15] Show that if the size of a minimum vertex cover in an n-vertex graph is vc, then the size of an independent set is at most $n - vc$. Use this to deduce that you found a maximum independent set in O.

We seek to show $is \leq n - vc$, where is is the maximum size of an independept set in our graph. This is equivalent to showing $vc \leq n - is$. Let IS be an independent set of size is. Since IS induces no edges, they must all touch the vertices, $V - IS$, hence $V - IS$ is a vertex cover of size $n - is$, thus the minimum size of a vertex cover, vc can be no larger.

[15] Bonus

Is the graph O from the previous problem planar? If so, produce a plane graph isomorphic to it. If not, justify your answer.