Department of Mathematics
Carnegie Mellon University

21-228 Discrete Mathematics, Summer II 2000: Exam 2

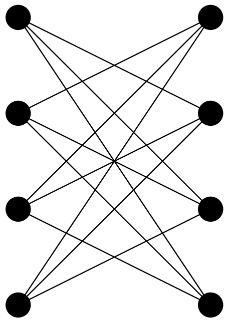
Name:

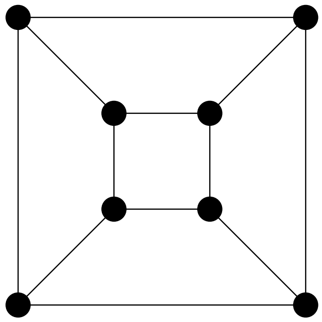
You have 90 minutes to complete this exam. Notes, books, and colloboration are prohibited.

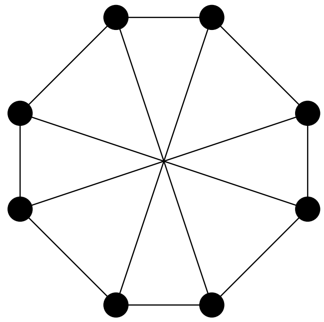
Problem	Score	Total
Isomorphism & planarity		20
Trees & matchings		30
Everything Eulerian		40
Everything else		60
Bonus		15
		150

[20] Isomorphism and planarity

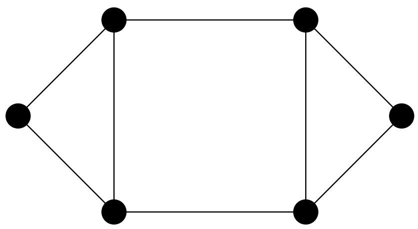
[10] Among the graphs below, which pairs are isomorphic? Justify your answer.

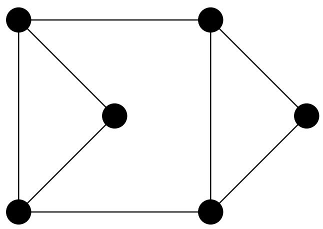






[10] The plane graphs below are isomorphic. Show that their duals are not isomorphic.





[30] Trees and matchings

A *spanning tree* of a graph $G = (V, E)$ is a subgraph that is a tree on the vertex set V. Recall that a *matching* of G is a subgraph of of maximum degree 1 (a collection of pairwise-disjoint edges). A matching M is *perfect* if for each vertex $u \in V$ there is some edge in M adjacent to u (every vertex is matched).

[**10**] Prove that the average degree of a vertex in a tree on n vertices is $2 - \frac{2}{n}$.

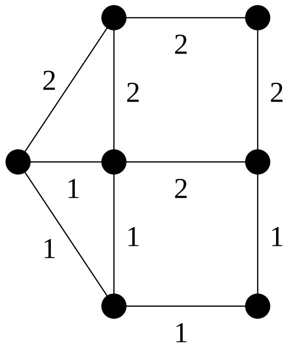
[**10**] Describe each spanning tree of C_n, the cycle on n vertices. How many are there?

[**10**] Describe each perfect matching of C_n. How many are there?

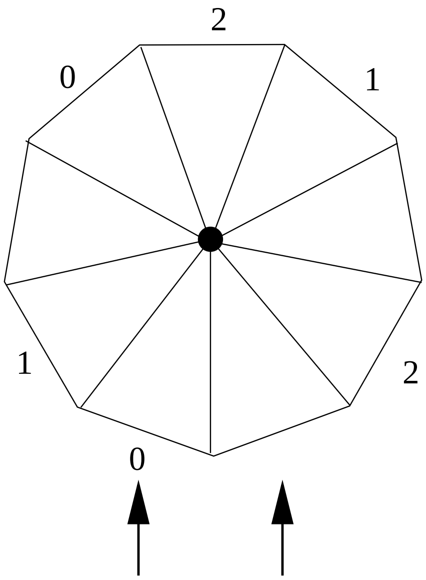
[40] Everything Eulerian

[**10**] Recall that given a connected graph $G = (V, E)$ and costs on the edges of E, the *Chinese Postman Problem* is that of finding a minimum-cost duplication of edges so that the resulting graph is Eulerian.

Solve the Chinese Postman Problem for the following graph.



[**10**] If the segments of the drum are labeled with elements from $\{0, 1, 2\}$ as shown below, can the remainder of the labels be filled in so that the position of the drum can be uniquely determined by reading the 2 indicated positions? Either produce a labeling, or briefly describe why the partial labeling below is inconsistent.

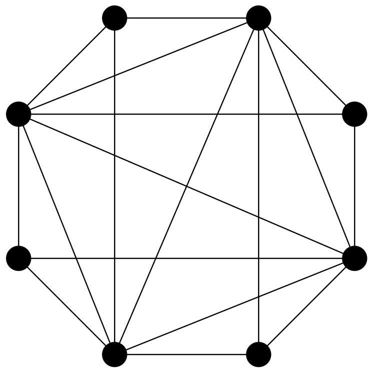


[**20**] Prove that a connected graph G in which u and v are the only vertices of odd degree has a u,v-Eulerian walk, a walk from u to v that traverses each edge exactly once. Extend this to show that a connected graph G which has exactly $2k$ odd vertices can be partioned into k walks, each between 2 odd vertices.

[Hint: A better hint lies on this page.]

[60] Everything else

A *vertex cover* in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that every edge in E is adjacent to some vertex in S. The set V is a trivial vertex cover. Recall that an *independent set* is a set of vertices $S \subseteq V$ such that $G[S]$, the subgraph of G induced by S, contains no edges.



[20] Find an independent set, matching, proper vertex coloring, and vertex cover in the graph O above, each of size 4.

[10] Prove that your proper coloring of O is optimal.

[Hint: How many colors does it take to color a complete subgraph?]

[15] Let m be the size of a maximum matching and vc the size of a minimum vertex cover in some graph G. Show that $m \leq vc$. Use this fact to deduce that the matching and vertex cover you found in O are indeed optimal.

[Hint: How many vertices does it take to cover the edges of a matching?]

[15] Show that if the size of a minimum vertex cover in an n-vertex graph is vc, then the size of an independent set is at most $n - vc$. Use this to deduce that you found a maximum independent set in O.

[Hint: What can you say about the vertices not in the vertex cover?]

[15] Bonus

Is the graph O from the previous problem planar? If so, produce a plane graph isomorphic to it. If not, justify your answer.