Department of Mathematics
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21-228 Discrete Mathematics, Summer II 2000: Exam 1 Solutions

Obligatory recurrence

(i) Let's count all valid sequences, $1, a_2, a_3, \ldots, a_{k-1}, n$ by considering the possibilities for a_{k-1}. The element a_{k-1} can only be one of $1, \ldots, n-1$ since our sequence must be strictly increasing:

a_{k-1}	Sequences
1	1: $1, n$
2	1: $1, 2, n$
$\vdots$	$\vdots$
i	s_i: take any sequence that ends in i tack on n
$\vdots$	$\vdots$
$n-1$	s_{n-1}

Summing all these up we get $s_n = 1 + s_2 + \ldots + s_{n-1}$. Our argument is correct since every valid sequence falls into **exactly** one of the cases in the table above.

(ii) Here's a nice inductive proof that some of you found. In the base case, $n = 2$ we do indeed have $2^0 = 1$ sequence, which is $1, 2$. Now assume $s_n = 2^{n-2}$, and let's show $s_{n+1} = 2^{n-1}$. We have

$$s_{n+1} = (1 + s_2 + \ldots + s_{n-1}) + s_n.$$

The stuff in the parentheses above is equal to s_n, so we can simplify to

$$s_{n+1} = 2s_n,$$

but now we apply the induction hypothesis to get $s_{n+1} = 2s_n = 2 \cdot 2^{n-2} = 2^{n-1}$. Nice and clean, eh?

Here's a more direct proof. Since the elements are in order all we have to know to construct a sequence is just which numbers go between 1 and n. If I tell you there are 0 elements between 1 and n you immediately know the only possible sequence is $1, n$. For example, if $n = 5$ and I tell you that 2 and 4 go in the middle, the only possible sequence is $1, 2, 4, 5$. Thus we can simply count all the ways of selecting the elements to go in the middle. We can select from the set $\{2, \ldots, n-1\}$, which has $n-2$ elements, so we get there are:

$$\binom{n-2}{0} + \binom{n-2}{1} + \cdots + \binom{n-2}{n-2} = 2^{n-2}$$

ways. We have derived and can derive the above identity by pluggin in $x = y = 1$ in the binomial theorem.

Poker anyone?

You can think of these many different ways, but you have to be sure that you're not over- or under-counting.

(i) We have 4 ways of picking the suit and $\binom{13}{5}$ ways of picking the ranks, for a total of $4\binom{13}{5}$ hands.

(ii) Let's pick the pair first. There are 13 choices for the rank of our pair and $\binom{4}{2}$ choices for the suits of the two cards. Now we have 50 cards left, but since we have exactly one pair, we can't pick any more of the rank of our pair, so we can only choose from 12 ranks. There are $\binom{12}{3}$ ways to pick the ranks of the remaining 3 cards, and $4 \cdot 4 \cdot 4$ ways of assigning suits to them. Putting this all together we get

$$13\binom{4}{2}\binom{12}{3}4^3$$

hands.

(iii) Let's pick our cards first and then count the orderings. We have $\binom{4}{2}\binom{4}{2}\binom{4}{1}$ ways of picking our aces, kings, and queen. Why isn't this overcounting? Now we have 5! ways of ordering the resulting hands, for a total of

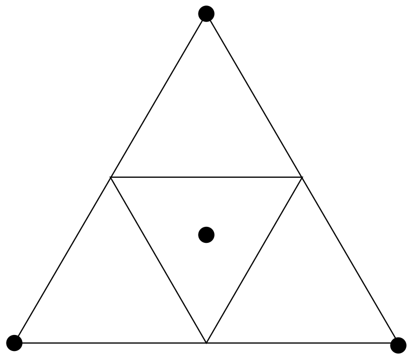
$$\binom{4}{2}\binom{4}{2}\binom{4}{1}5!$$

hands.

(iv) Let's just take all hands and subtract the bad ones (ones with no clubs, or no diamonds, ...). We have $\binom{52}{5}$ hands total and $\binom{39}{5}$ hands that don't contain any of a particular suit. So we subtract these out to get $\binom{52}{5} - 4\binom{39}{5}$. Are we done? Well what about a hand that has no clubs and no diamonds? We've subtracted them out at least twice... We continue with the inclusion-exclusion reasoning to get

$$\binom{52}{5} - 4\binom{39}{5} + \binom{4}{2}\binom{26}{5} - \binom{4}{3}\binom{13}{5} + \binom{4}{4}0.$$

Where're the pigeons?



(i)-(ii) 4 such points are indicated in the diagram above. If the large equilateral triangle has side length 1 unit, then each of the 4 smaller equilateral trianlges have a side length of $\frac{1}{2}$ unit, so any two points in one of the 4 triangles would have to be within $\frac{1}{2}$ unit of each other. If we place 5 points in the triangle above, 2 would fall in the same subtriangle, making them too close to each other, so we know 4 is the best we can do.

Plug and chug, baby

(i) We know the roots of our characteristic equation have to be 2 and 5, so our equation is $(r-2)(r-5) = r^2 - 7r + 10 = 0$, giving us the recurrence $a_n = 7a_{n-1} - 10a_{n-2}$. We get $a_0 = 3 - 1 = 2$ and $a_1 = 2 \cdot 3 - 5 = 1$.

(ii) Our characteristic equation is $r^2 - 2r - 1 = 0$, hence our roots are

$$r_1 = \frac{2 - \sqrt{4 - 4(-1)}}{2} = \frac{2 + 2\sqrt{2}}{2} = 1 + \sqrt{2},$$

and

$$r_2 = 1 - \sqrt{2}.$$

Our general solution is

$$b_n = c_1(1 + \sqrt{2})^n + c_2(1 - \sqrt{2})^n.$$

The condition $b_0 = 0$ gives us $c_1 = -c_2$. Plugging this into the equation derived from $b_1 = 1$ we get

$$c_1(1 + \sqrt{2}) - c_1(1 - \sqrt{2}) = 1,$$

which gives us $c_1 = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} = -c_2$. Our solution is

$$\frac{\sqrt{2}}{4}(1 + \sqrt{2})^n - \frac{\sqrt{2}}{4}(1 - \sqrt{2})^n.$$

(iii) We first find a solution to the homogeneous recurrence $h_n = 2h_{n-1}$. Our characteristic equation, $r = 2$, solves itself giving us

$$h_n = c \cdot 2^n,$$

where we'll solve c in terms of c_0 later.

Now we need just find one particular solution, p_n that satisfies $p_n = 2p_{n-1} + n + 1$ for $n \geq 1$. What we mean by a particular solution is that it won't have any variables other than n. Since $p_n - 2p_{n-1} = n + 1$, a reasonable guess is that $p_n = an + b$. Let's try this:

$$\begin{aligned} p_n - 2p_{n-1} &= n + 1 \Rightarrow \\ (an + b) - 2(a(n-1) + b) &= n + 1 \Rightarrow \\ -an + (2a - b) &= n + 1, \end{aligned}$$

hence $a = -1$ and $b = -3$. Thus

$$p_n = -n - 3,$$

and

$$c_n = h_n + p_n = c \cdot 2^n - n - 3.$$

When $n = 0$ we have $c_0 = c - 3$, so $c = c_0 + 3$. Breathing a sigh of relief,

$$c_n = (c_0 + 3)2^n - n - 3.$$