Department of Mathematics
Carnegie Mellon University

21-228 Discrete Mathematics, Summer II 2000: Exam 1

Name:

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Obligatory recurrence (40 points)

(20)(i) Show that the recurrence for the number of strictly increasing sequences of positive integers that have 1 as their first term and n as their last term where n is a positive integer is

$$s_n = 1 + s_{n-1} + s_{n-2} + ... + s_2,\ n \geq 3;\ s_2 = 1.$$

Just to clarify we are interested in sequences $a_1, a_2, ..., a_k$ where $a_1 = 1$, $a_k = n$, and $a_j < a_{j+1}$ for $j = 1, 2, ..., k-1$. We have $s_4 = 4$:

$$\begin{array}{l} 1,4 \\ 1,2,4 \\ 1,3,4 \\ 1,2,3,4 \end{array}$$

[Hint: What happens when you remove 'n' from each such sequence?]

(20)(ii) Show that $s_n = 2^{n-2}$ either by induction, applying the useful identity $\sum_{i=0}^{l} 2^i = 2^{l+1} - 1$, or otherwise.

[Hint if you don't use induction: Since the elements between 1 and n are in order, all that matters is the numbers chosen.]

Poker anyone? (40 points)

Assume the questions are addressing a 5-card poker hand chosen from a standard deck of 52 cards.

(8)(i) How many hands are a flush (all cards of the same suit)?

(8)(ii) How many hands contain exactly one pair?

(10)(iii) If the order of the cards drawn matters, in how many ways can a hand with 2 Aces, 2 Kings, and 1 Queen be drawn?

(14)(iv) How many hands with at least 1 card of each suit?

Where're the pigeons? (30 points)

(10)(i) Show a way to place 4 points in a an equilateral triangle of side length 1 unit so that no point is within $\frac{1}{2}$ unit of any other.

(20)(ii) Show that this is best possible; that is, in an equilateral triangle of side length 1 unit at most 4 points may be placed so that no point is within $\frac{1}{2}$ unit of any another.

[Hint: Two points in an equilateral triangle of side length of $\frac{1}{2}$ unit are always within $\frac{1}{2}$ unit of each other]

Plug and chug, baby (40 points)

(10)(i) Find a second order linear recurrence satisfied by

$$a_n = 3 \cdot 2^n - 5^n,\ n \geq 0.$$

Don't forget initial conditions.

(10)(ii) Solve.

$$b_n = 2b_{n-1} + b_{n-2},\ n \geq 2;\ b_0 = 0,\ b_1 = 1$$

(20)(iii) Find a general solution (one in terms of c_0) for

$$c_n = 2c_{n-1} + n + 1,\ n \geq 1.$$