21-260 Midterm

Problem 1 Graph the Directional Fields described by the following equations.

a) $y' = xy$

b) $y' = (y-1)(y+2)$

Problem 2 Find the integrating factor for the following differential equations. Then use it to solve for $y(t)$.

a) $y' - y = 2e^{2t}$

b) $(\sin t)y' + (\cos t)y = e^t$

Solution 2

a) Let $u(t)$ be the integrating factor.

$$u(t) = e^{\int -1\, dt} = e^{-t}$$

Then the solution is

$$y(t) = \frac{1}{u(t)}(\int u(t)\, 2e^{2t}\, dt + c) = \frac{1}{e^{-t}}(\int 2e^{t}\, dt + c) = \frac{1}{e^{-t}}(2e^t + c) = 2e^{2t} + c\, e^t$$

b) We can rewrite this equation in the canonical form to get

$$y' + \frac{\cos t}{\sin t}\, y = \frac{e^t}{\sin t}$$

Let $u(t)$ be the integrating factor.

$$u(t) = e^{\int \frac{\cos t}{\sin t}\, dt} = e^{\ln|\sin t|} = \sin t$$

Then the solution is

$$y(t) = \frac{1}{\sin t}(\int \sin t \frac{e^t}{\sin t}\, dt + c) = \frac{1}{\sin t}(\int e^t\, dt + c) = \frac{1}{\sin t}(e^t + c) = \frac{e^t + c}{\sin t}$$

Problem 3 Solve the following differential equation:

$$y' = \frac{1 + 3x^2}{3y^2 - 6y}$$

Solution 3 The key here is to use separation of variables. So we rewrite the equation to get

$$(3y^2 - 6y)\frac{dy}{dx} = 1 + 3x^2$$

Integrating both sides gives us

$$\int 3y^2 - 6y\, dt = \int 1 + 3x^2\, dt$$

$$y^3 - 3y^2 = x + x^3 + c$$

Note that this is the final form of the solution.

Problem 4 Solve the following initial value problems and state where the solution is valid.

a) $(2x - y) + (2y - x)y' = 0 \qquad y(1) = 3$

b) $(9x + \frac{y-1}{x}) - (\frac{4y}{x} - 1)y' = 0 \qquad y(1) = 0$

Solution 4

a) Clearly we have to use the method of exact equations. So let $M = 2x - y$ and $N = 2y - x$. Checking to see if this equation is exact we get

$$M_y = -1 = N_x \,.$$

The general solution $\phi(x, y) = c$ where $\phi(x, y) = Q(x, y) + h(y)$. Here

$$Q(x, y) = \int M(x, y)dx = \int 2x - y\, dx = x^2 - xy + c$$

and

$$h'(y) = N(x, y) - Q_y = 2y - x - (-x) = 2y \,.$$

So integrating this we get

$$h(y) = \int 2y\, dy = y^2 + c \,.$$

Combining these two we get

$$\phi(x, y) = Q(x, y) + h(y) = x^2 + y^2 - xy = c$$

Solving for initial condition we get

$$1^2 + 3^2 - 1 \cdot 3 = 7 = c \,.$$

So the final answer is

$$x^2 + y^2 - xy = 7 \,.$$

b) In this problem $M = (9x + \frac{y-1}{x})$ and $N = 1 - \frac{4y}{x}$. So we first check to see whether this is exact.

$$M_y = \frac{1}{x} \neq \frac{4y}{x^2} = N_x$$

So we need to find an integrating factor. First suppose that $u(x, y)$ is only a function of x. Then

$$\frac{1}{u(x)}\frac{du}{dx} = \frac{M_y - N_x}{N} = \frac{1/x - (4y/x^2)}{1 - 4y/x} = \frac{1}{x} \,.$$

Integrating and solving for u we get $u(x) = x$.

Now multiplying the original equation by $u(x)$ we get

$$(9x^2 + y - 1) + (x - 4y)y' = 0\,.$$

So

$$Q(x,y) = \int M(x,y)dx = \int 9x^2 + y - 1 \; dx = 3x^3 + xy - x + c$$

and

$$h'(y) = N(x,y) - Q_y = x - 4y - (x) = -4y\,.$$

So

$$h(y) = \int -4y\,dy = -2y^2 + c\,.$$

Combining these two we get

$$\phi(x,y) = Q(x,y) + h(y) = 3x^3 + xy - x - 2y^2 = c$$

Solving for initial condition we get

$$3 - 1 = 2 = c\,.$$

So the final answer is

$$3x^3 + xy - x - 2y^2 = 2\,.$$

Problem 5 Solve the following initial value problems and state where the solution is valid.

a) $y'' + 2y' + y = 0 \quad y(0) = 0 \quad y('0) = 1$

b) $y'' + 2y' + y = \sin t \quad y(0) = 0 \quad y'(0) = 1$

c) $y'' + 2y' + y = \sin t + \cos t \quad y(0) = 0 \quad y'(0) = 1$

Solution 5

a) The corresponding characteristic equation is

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0\,.$$

So the general solution is

$$y = c_1 e^{-t} + c_2 t e^{-t}\,.$$

Plugging in the initial conditions we get $0 = c_1$ from the first condition and $c_2 = 1$ from the second condition. So the final solution is $y(t) = te^{-t}$.

b) The same characteristic equation applies here, however the method of undetermined coefficient tells us that we should try the particular solution $y(t) = a\cos t$. Plugging this into the differential equation we get

$$-a\cos t + 2(-a\sin t) + a\cos t = \sin t$$

which gives us $a = -1/2$. The the general solution is

$$y(t) = c_1 e^{-t} + c_2 t e^{-t} - 1/2\sin t\,.$$

Plugging in the two initial conditions we get $c_1 = 0$ and $c_2 = 3/2$. So the final answer is

$$y(t) = 3/2te^{-t} - 1/2\sin t\,.$$

c) Using the fact that all three differential equation are linear we know that we have to only solve for $y'' + 2y' + y = \cos t$ and add that solution to our previous answers. So let $y(t) = a\sin t$. The

$$-a\sin t + 2(a\cos t) + a\sin t = \cos t\,.$$

So $a = 1/2$. The general solution is of the form $y(t) = c_1 e^{-t} + c_2 t e^{-t} - 1/2\sin t + 1/2\cos t$. Solving for the constants using initial conditions we get

$$y(t) = -1/2e^{-t} + 1/2te^{-t} - 1/2\sin t + 1/2\cos t\,.$$

Problem 6 State the longest interval in which the following differential equation has a unique twice differentiable solution.

$$(x-3)y'' + xy' + \ln|x|\,y = 0 \quad y(1) = 0 \quad y'(1) = 1$$

Solution By theorem 3.2.1 we need to find an interval which contains the initial condition and where $p(t)$, $q(t)$ and $g(t)$ are continuous. So rewriting the equation in canonical form, we get

$$y'' + \frac{x}{x-3}y' + \frac{\ln|x|}{x-3}\,y = 0\,.$$

Note that $p(t)$ and $q(t)$ are continuous in the interval $(0, 3)$ and the initial condition lies there.

Problem 7 Find the Wronskian corresponding to the following differential equation. What does the Wronskian tell you about the about the solution?

$$\lambda^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

Solution 7 The Wronskian of this equation is

$$W(x) = e^{\int -p(x)\,dx} = e^{\int -\frac{x}{\lambda^2}\,dx} = e^{-x^2/2\lambda^2}\,.$$

Which is non zero everywhere. So the solution to this differential equation exists for all values of x, as long as $\lambda \neq$ and thus all initial conditions. When $\lambda = 0$ this reduces to a first order equation for which the solution exists for $x > 0$.

Problem 8 Calculate the Laplace transform of the following functions.
a) $\sin a\, t$
b) $\delta(t - \pi/4)\sin t$
Solution 8
a)

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st}\sin at\,dt = \frac{-1}{a}e^{-st}\cos at|_0^\infty - \frac{s}{a}\int_0^\infty e^{-st}\cos at\,dt$$

$$= \frac{1}{a} - \frac{s}{a}(\frac{1}{a}e^{-st}\sin at|_0^\infty + \frac{s}{a}\int_0^\infty e^{-st}\sin at\,dt) = \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin at\,dt$$

Rearranging the two sides we get

$$(1 + \frac{s^2}{a^2})\mathcal{L}\{\sin at\} = \frac{1}{a}$$

which leads to

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}\,.$$

b) The solution for this follows directly from the definition of the Dirac Delta function

$$\mathcal{L}\{\delta(t - \pi/4)\sin t\} = \int_0^\infty e^{-st}\delta(t - \pi/4)\sin t\,dt = e^{-s\pi/4}\sin \pi/4 = \frac{e^{-s\pi/4}}{\sqrt{2}}\,.$$

Problem 9 Use Laplace tasnforms to solve the following initial value problems.
a) $y'' + 2y' + y = 4e^{-t} \quad y(0) = 2 \quad y'(0) = -1$
b) $y'' + y = u_{\pi/2}(t) + 3\delta(t - 3\pi/2) - u_{2\pi}(t) \quad y(0) = 0 \quad y'(0) = 0$
Solution 9
a) Taking Laplace transform of both sides we get

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) = \frac{4}{s+1}\,.$$

$$(s^2 + 2s + 1)Y(s) - 2s - 3 = \frac{4}{s+1}$$

$$Y(s) = \frac{4}{(s+1)^3} + \frac{2}{s+1} + \frac{1}{(s+1)^2}$$

Now using the fact that $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$.

$$y(t) = 2t^2 e^{-t} + te^{-t} + 2e^{-t}\,.$$

b) Taking Laplace transform of both sides we get

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{e^{-\pi s/2}}{s} + 3e^{-3\pi s/2} - \frac{e^{-2\pi s}}{s}$$

$$Y(s) = e^{-\pi s/2}(\frac{1}{s} - \frac{1}{s^2+1}) + \frac{3e^{-3\pi s/2}}{s^2+1} - e^{-2\pi s}(\frac{1}{s} - \frac{1}{s^2+1})\,.$$

The relevant equations here are $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$, $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$ and $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$. Using these we get

$$y(t) = u_{\pi/2}(t)(1 - \cos(t - \pi/2)) + 3u_{3\pi/2}(t)\sin(t - 3\pi/2) - u_{2\pi}(1 - \cos(t - 2\pi))\,.$$

Problem 10 Express the solution to the following differential equation in terms of a convolution integral.

$$y'' + 4y' + y = g(t) \quad y(0) = 2 \quad y'(0) = -3$$

Solution 10 Taking Laplace transforms of both sides we get

$$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + Y(s) = G(s)$$

$$(s^2 + 4s + 1)Y(s) - 2s - 5 = G(s)$$

$$Y(s) = \frac{2(s+2)}{(s+2)^2 - 3} + \frac{1}{(s+2)^2 - 3} + \frac{G(s))}{(s+2)^2 - 3}$$

Since $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$ and $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$ we get

$$y(t) = 2e^{-2t}\cos(\sqrt{3}it) + \frac{e^{-2t}}{\sqrt{3}i}\sin(\sqrt{3}it) + \int_0^t g(t-\tau)\frac{e^{-2\tau}}{\sqrt{3}i}\sin(\sqrt{3}i\tau)\,d\tau\,.$$