Which Rectangular Chessboards Have a Knight's Tour?

ALLEN J. SCHWENK*
Western Michigan University
Kalamazoo, MI 49008

Problems involving the search for Hamiltonian cycles are popular in undergraduate discrete mathematics courses. A few textbooks introduce the intriguing puzzle of searching for spanning tours by a knight on various rectangular chessboards. This area provides a down-to-earth collection of problems that illustrates the idea of a Hamiltonian cycle. The problems are challenging enough to require thoughtful solutions, and yet, at least for small boards, manageable enough so that students can succeed in finding tours on some boards and in showing that they are impossible on others. It also gives the instructor a chance to prove the nonexistence of tours on an infinite family of boards by an elegant (though well-known) parity argument. Certainly any curious student must wonder precisely which size boards do admit knight's tours and which do not. Chartrand [2] ignores this natural question, while Wilson and Watkins [7] report that the question was fully resolved by Euler in 1759 and 12 years later (independently) by Vandermonde. Similarly, Berge [1] introduces the problem, mentions some of the history, and then immediately drops it. Dudeney [3] also provides a sketchy history. Rouse Ball and Coxeter [6] provide a 10-page treatment of the problem without ever mentioning which size boards can in fact be toured. A recent research article by Eggleton and Eid [4] focuses on "open" tours for which the knight need not return to his starting square. They even extend the problem to infinite boards of various types, leading to intriguing questions about the existence of spanning one-way and two-way infinite paths. But their discussion of the original knight's tour problem only goes into detail on the well-known odd order case and on the family of $3 \times n$ boards where they report a private communication claiming (erroneously) that Hamiltonian cycles exist if and only if $n \geqslant 8$ and n is even. We shall show that the correct version is $n \geqslant 10$ and n is even. The universal avoidance of reporting the definitive solution creates the impression that it must be beyond the undergraduate level. Presumably, it is difficult to describe the sizes that admit a tour, harder still to actually construct these tours, and heaven knows what it takes to show that all other sizes really are impossible. The 200-year-old references to the literature are incomplete and intimidating. I don't know how to find these ancient volumes. My students wouldn't even consider trying.

The purpose of this article is to show that the full solution of the knight's tour problem is quite brief and entirely accessible to beginning students. In the process, the student will see a new use of parity to show impossibility in one case and a rather unusual instance of proof by induction that requires nine specific cases in order to anchor the induction.

We begin with a careful definition of the problem. An $m \times n$ chessboard is an array with square cells arranged in m rows and n columns. The standard chessboard is 8×8. For convenience we shall assume $m \leqslant n$. We label the cells (i, j) counting from the upper left corner in matrix fashion. Now a legal knight move is the result of moving two cells horizontally or vertically and then turning and moving one cell in

*Research supported in part by the Office of Naval Research, Contract N0014-91-J-1364.

the perpendicular direction. Thus, if we start at cell (i, j) we can complete the move on one of eight cells: $(i \pm 2, j \pm 1)$ or $(i \pm 1, j \pm 2)$. Of course if we are too close to the border of the board some of these choices may not exist. The knight's tour question is usually posed in this form:

Problem. On which $m \times n$ boards can a knight make successive legal knight moves, visit every cell exactly once, and conclude by returning to its starting cell?

There is also a version of the problem seeking "open tours" where the knight is not required to return to his starting position. The open tour problem can be solved by the same methods as the more common "closed tour" problem; we shall leave it as a challenge for the interested reader. The first step is to convert the problem to a question about certain graphs. We define a graph $G(m, n)$ on mn vertices by replacing each cell of the board by a vertex and then joining two vertices by an edge if they are separated by a knight's move. This is illustrated for a 3×6 board in FIGURE 1. A knight can tour the $m \times n$ board if and only if there exists a cycle containing all the vertices in the resulting graph. Such a cycle is called a Hamiltonian cycle, named after William R. Hamilton who marketed a puzzle called *A Voyage Round the World* based on this concept in 1859. Accounts of Hamilton's puzzle can be found in [2, 6, 7]. The customary alternating white and black coloring of the chessboard is preserved in the white and black vertices of the graph. We set vertex (i,j) to be white if $i+j$ is even and black if $i+j$ is odd. It is easy to see that every edge in the graph joins vertices of opposite colors. Such a graph is called *bipartite*, or for brevity, a *bigraph*. Since the colors must alternate in any cycle, the cycle must have an even number of vertices. We have just proved one of the first theorems on bipartite graphs, namely, all cycles must be even.

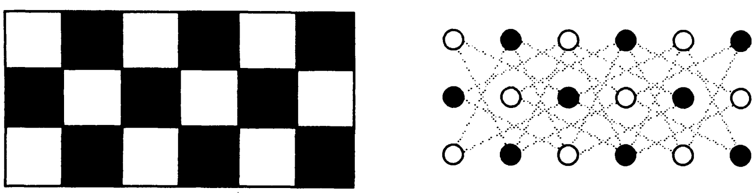

FIGURE 1
Conversion of the 3×6 chessboard into the graph $G(3, 6)$.

We can now state the conditions that determine which chessboards have a knight's tour.

THEOREM. *An $m \times n$ chessboard with $m \leqslant n$ has a knight's tour unless one or more of these three conditions holds*:
(a) *m and n are both odd*;
(b) *$m = 1$, 2, or 4; or*
(c) *$m = 3$ and $n = 4$, 6, or 8.*

Proof. We begin by showing why conditions (a), (b), and (c) must be excluded. Then we shall show how to construct a tour on every other board.

When m and n are both odd, so is the order, mn, of our graph. But we have already observed that every cycle must be even, and so no Hamiltonian cycle can exist.

For $m = 1$ or 2, it is clear that the board is not wide enough to permit a tour. Indeed, cell $(1,1)$ doesn't even have two available edges to be used in the cycle. For $m = 4$, the impossibility is more subtle. We present here the proof discovered by

Louis Pósa as a teenager and reported in the classic book of Ross Honsberger [5]. Assume that we have found a Hamiltonian cycle $v_1v_2\dots v_{4n}v_1$. Let us recolor the vertices red and blue, with every vertex in rows 1 and 4 red and every vertex in rows 2 and 3 blue. This coloring no longer serves as the bipartition for the graph since some blue vertices are adjacent to other blue vertices, for example, (2, 1) and (3, 3). However, every red vertex is adjacent only to blue vertices. Thus, in a presumed Hamiltonian cycle, the red vertices must always be separated by blue vertices. Since we have $2n$ vertices of each color, the red and blue vertices must alternate around the cycle. Now starting at $v_1 = (1, 1)$, we can conclude that all the vertices in odd positions on the cycle, v_{2k+1}, are red. But from the original black and white coloring we can conclude equally well that all the vertices v_{2k+1} are also white. Thus all red vertices are white vertices, but this contradicts the different pattern chosen for the two colorings. We conclude that no Hamiltonian cycle is possible.

To analyze condition (c) we introduce certain graphical concepts. The 3×4 board has already been excluded in the preceding paragraph. When we remove a vertex v from a graph G we also remove all edges incident with v. For any G having a Hamiltonian cycle, it is clear that removing any set of k vertices can leave at most k connected components. Since removing vertices (1, 3) and (3, 3) from $G(3,6)$ leaves three components, we must conclude that no Hamiltonian cycle exists for the 3×6 board. Now it happens that in $G(3,8)$ vertices (1, 1), (2, 1), (3, 1), (2, 2), (1, 8), (2, 8), (3, 8), and (2, 7) all have degree two, forcing us to include in a presumed Hamiltonian cycle the 16 edges shown in Figure 2. These edges form six paths that must lie within the Hamiltonian cycle. We also consider the two vertices missed by all six paths, namely (2, 4) and (2, 5) as trivial paths. We define a new graph $G^*(3,8)$ derived from these eight paths by letting one new vertex stand for each of these eight paths and joining two of these new vertices i and j whenever there is an edge in $G(3,8)$ joining an end of path i to an end of path j. Now a Hamiltonian cycle in the original $G(3,8)$ must force a corresponding Hamiltonian cycle to be present in $G^*(3,8)$, although the converse need not be true. But $G^*(3,8)$ has two vertices of degree three whose removal leaves three components. Therefore, neither $G^*(3,8)$ nor $G(3,8)$ can have a Hamiltonian cycle.

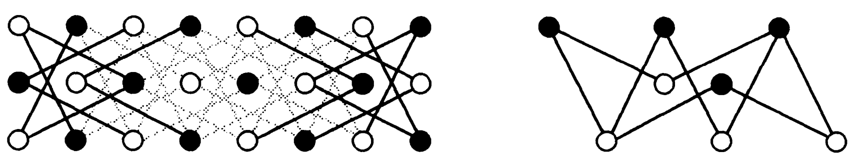

FIGURE 2

Sixteen edges that must belong to any Hamiltonian cycle of $G(3,8)$ and the resulting derived graph $G^*(3,8)$.

This completes the list of excluded sizes. Every other board has a Hamiltonian cycle, but how can we hope to construct all the necessary tours? The key is to develop a method that allows us to build new tours from smaller tours. In the following lemma it is convenient to dispense with the convention that $m \leqslant n$. This lemma allows us to add 4 columns to a successful tour, provided 10 particular edges belong to the tour. Actually, our extension methods may require one, two, or four particular edges to be present. The union of these sets gives five prescribed edges. But in order to be free to extend by either four columns or four rows, we require the presence of 10 specified edges.

LEMMA. *If $G(m, n)$ has a Hamiltonian cycle that includes the* 10 *edges*

$$(1, n-1)\text{–}(3, n) \quad (m-2, n-1)\text{–}(m, n) \quad (m-1, 1)\text{–}(m, 3) \quad (m-1, n-2)\text{–}(m, n)$$
$$(4, n-1)\text{–}(2, n) \quad (1, n)\text{–}(3, n-1) \quad (m-2, n)\text{–}(m, n-1) \quad (m, 1)\text{–}(m-1, 3)$$
$$(m, n-2)\text{–}(m-1, n) \quad (m, 2)\text{–}(m-1, 4),$$

then $G(m, n+4)$ also has a Hamiltonian cycle including the corresponding 10 *edges*

$$(1, n+3)\text{–}(3, n+4) \quad (m-2, n+3)\text{–}(m, n+4) \quad (m-1, 1)\text{–}(m, 3)$$
$$(m-1, n+2)\text{–}(m, n+4) \quad (4, n+3)\text{–}(2, n+4) \quad (1, n+4)\text{–}(3, n+3)$$
$$(m-2, n+4)\text{–}(m, n+3) \quad (m, 1)\text{–}(m-1, 3) \quad (m, n+2)\text{–}(m-1, n+4)$$
$$(m, 2)\text{–}(m-1, 4).$$

Proof. The 10 required edges are displayed visually in FIGURE 3. For $m = 3$, these "10 edges" degenerate into a set of seven. For all values of m and n, four of the 10 required edges (specifically the edges $(1, n)$–$(3, n-1)$, $(m, 1)$–$(m-1, 3)$, $(m-2, n-1)$–(m, n), and $(m-1, n-2)$–(m, n) that all lead into corner cells) are already forced to obtain any Hamiltonian cycle. Thus, the additional hypothesis needed to facilitate the induction is not as restrictive as it may at first appear. To add four columns to any Hamiltonian cycle in $G(3, n)$ that contains the critical seven edges, we place a certain 3×4 array with a spanning path along side $G(3, n)$, delete edge $(1, n-1)$–$(3, n)$ from the cycle, and insert edges $(1, n-1)$–$(2, n+1)$ and $(3, n)$–$(1, n+1)$ in order to incorporate the path into the cycle. FIGURE 4 shows the extension of a Hamiltonian cycle in $G(3, 10)$ to one in $G(3, 14)$ to illustrate this construction. The new Hamiltonian cycle also contains the prescribed seven edges, so it too can be used for further extensions.

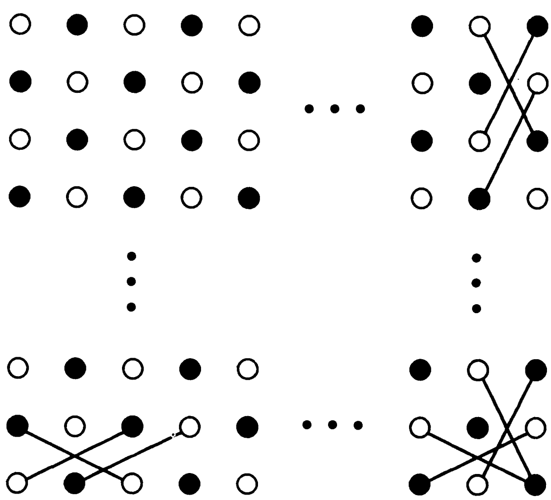

FIGURE 3
The ten edges required for the proposed induction.

For $m \geqslant 5$, we use an $m \times 4$ array $H(m, 4)$ that is obtained from $G(m, 4)$ by deleting all edges joining column two to column three and all edges joining vertices two columns apart except those joining vertices in rows 1 and 2 and those joining vertices in rows $m-1$ and m. The remaining graph $H(m, 4)$ is regular of degree 2, that is, every vertex has degree 2. Its edges form cycles that wrap around the board hugging the outside border as closely as possible, but it is not a single cycle. It is easy to see that $H(m, 4)$ has a pair of $2m$-cycles when m is odd and four m-cycles when m

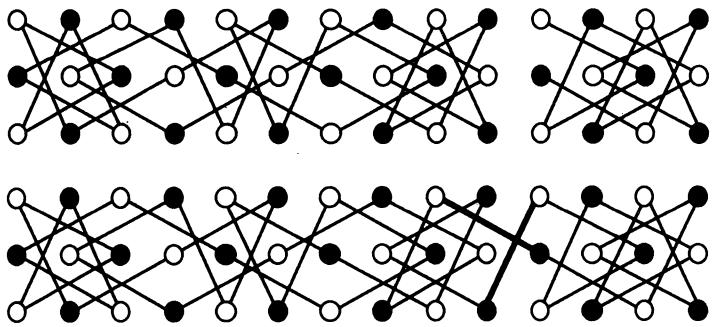

FIGURE 4
Extension of a Hamiltonian cycle in $G(3, n)$ to one in $G(3, n+4)$ for $n = 10$.

is even. Since this fact is critical for the construction we are developing, we shall prove it by induction.

FIGURE 5 displays one of the cycles in $H(5,4)$. Its mate is formed by the reflection about the vertical axis through the center of the board. The additional two rows of vertices at the bottom suggest how the cycle shown is extended in $H(7,4)$. Specifically, we delete edges $(5,1)$–$(4,3)$ and $(5,2)$–$(4,4)$ and insert edges $(5,1)$–$(7,2)$, $(7,2)$–$(6,4)$, $(6,4)$–$(4,3)$, $(5,2)$–$(7,1)$, $(7,1)$–$(6,3)$, and $(6,3)$–$(4,4)$. Repeating this extension displays the paired-cycle structure in $H(m,4)$ whenever m is odd.

Similarly, FIGURE 6 displays two of the four cycles in $H(6,4)$. The other two mates are found by using a vertical reflection. Analogous to the odd case, we extend the cycles shown by deleting edges $(6,1)$–$(5,3)$ and $(6,2)$–$(5,4)$ and inserting edges $(6,1)$–$(8,2)$, $(8,2)$–$(7,4)$, $(7,4)$–$(5,3)$, $(6,2)$–$(8,1)$, $(8,1)$–$(7,3)$, and $(7,3)$–$(5,4)$. Repeating this extension displays the four cycle structure in $H(m,4)$ whenever m is even.

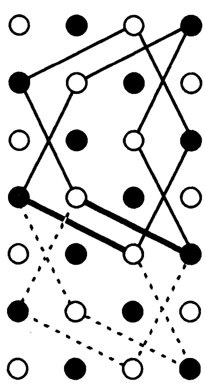

FIGURE 5
One of the cycles in $H(5,4)$ and its extension to $H(7,4)$.

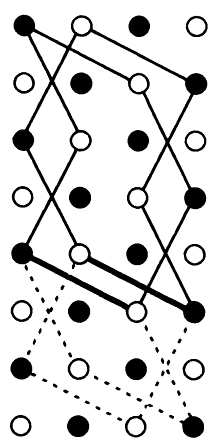

FIGURE 6
Two of the cycles in $H(6,4)$ and its extension to $H(8,4)$.

To extend a Hamiltonian cycle in $G(m,n)$ with m odd to one $G(m, n+4)$, we place $H(m,4)$ along side of $G(m,n)$ as shown in FIGURE 7. We remove two edges $(1,n)$–$(3,n-1)$ and $(2,n)$–$(4,n-1)$ from the Hamiltonian cycle, and two edges $(1,n+2)$–$(3,n+1)$ and $(2,n+2)$–$(4,n+1)$ from $H(m,4)$, and then insert four edges $(1,n)$–$(2,n+2)$, $(2,n)$–$(1,n+2)$, $(3,n-1)$–$(4,n+1)$, and $(4,n-1)$–$(3,n+1)$. This has the effect of incorporating the two cycles of $H(m,4)$ into the given Hamiltonian cycle to create a new Hamiltonian cycle in $G(m,n+4)$. The new cycle contains the prescribed 10 edges. The extension of $G(5,6)$ to $G(5,10)$ is illustrated in FIGURE 7.

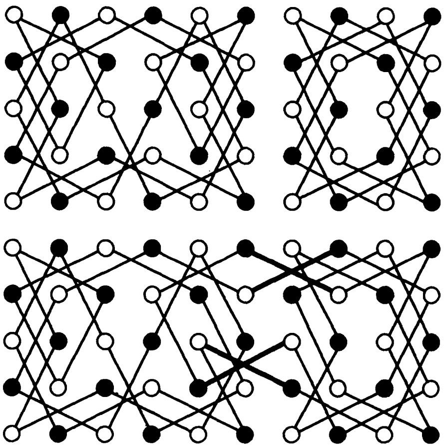

FIGURE 7
Extension of a Hamiltonian cycle in $G(5,n)$ to one in $G(5,n+4)$ for $n=6$.

Similarly, for m even as in FIGURE 8, we can incorporate $H(m,4)$ into a Hamiltonian cycle of $G(m,n)$ by first removing the four edges

$$(1,n-1)\text{–}(3,n),\ (1,n)\text{–}(3,n-1),\ (m-2,n-1)\text{–}(m,n),\quad \text{and}$$
$$(m-2,n)\text{–}(m,n-1)$$

from the Hamiltonian cycle, then removing the four edges

$$(2,n+1)\text{–}(4,n+2),\ (2,n+2)\text{–}(4,n+1),\ (m-3,n+1)\text{–}(m-1,n+2),\quad \text{and}$$
$$(m-3,n+2)\text{–}(m-1,n+1)$$

from $H(m,4)$, and finally inserting the eight edges

$$(1,n-1)\text{–}(2,n+1),\ (1,n)\text{–}(2,n+2),\ (3,n-1)\text{–}(4,n+1),\ (3,n)\text{–}(4,n+2),$$
$$(m-2,n-1)\text{–}(m-3,n+1),\ (m-2,n)\text{–}(m-3,n+2),$$
$$(m,n-1)\text{–}(m-1,n+1),\quad \text{and}\quad (m,n)\text{–}(m-1,n+2).$$

Again, the new cycle contains the prescribed 10 edges. The extension of $G(6,6)$ to $G(6,10)$ is illustrated in FIGURE 8. This completes the proof of the lemma.

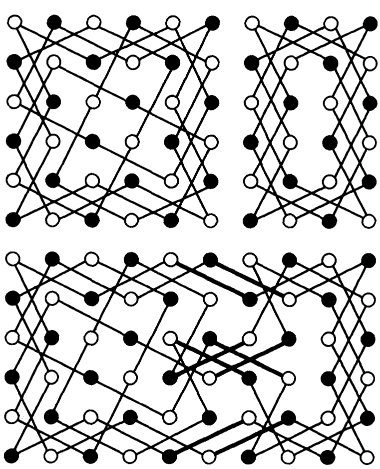

FIGURE 8
Extension of a Hamiltonian cycle in $G(6,n)$ to one in $G(6,n+4)$ for $n=6$.

But now we must complete the proof of the theorem. The lemma can be used to add four columns or four rows to a known solution. Thus, we can construct solutions for $G(m,n)$ provided we have a collection of starting cases for each possible modulo class pair $[i,j]$ where both i and j are taken modulo 4. Thus, it might seem that we would need 16 instances to serve as the base of our construction. But by flipping a rectangle over about its main diagonal we can interchange i and j, reducing the count to 10 instances. Moreover, recall that condition (a) forbids odd ordered boards, excluding classes [1,1], [1,3], and [3,3]. This leaves just seven classes. Considering the forbidden values of $m = 1$, 2, and 4, the smallest possible members of these seven classes are 3×6, 3×8, 5×6, 5×8, 6×6, 6×8, and 8×8. But condition (c) excludes size 3×6. To replace it and be able to generate all other orders in the same class, we must include both 3×10 and 7×6. Similarly, the impossible order 3×8 forces us to include both 3×12 and 7×8. Thus, there are nine specific instances whose Hamiltonian cycles must be constructed individually before the lemma can be used to finish the job by induction. I have no particular method for generating these nine solutions, but whenever possible I have tried to select solutions that have pleasing symmetry or near symmetry. All nine are collected in FIGURE 9. I couldn't resist the urge to seek the most compact arrangement of the nine solutions into a single drawing.

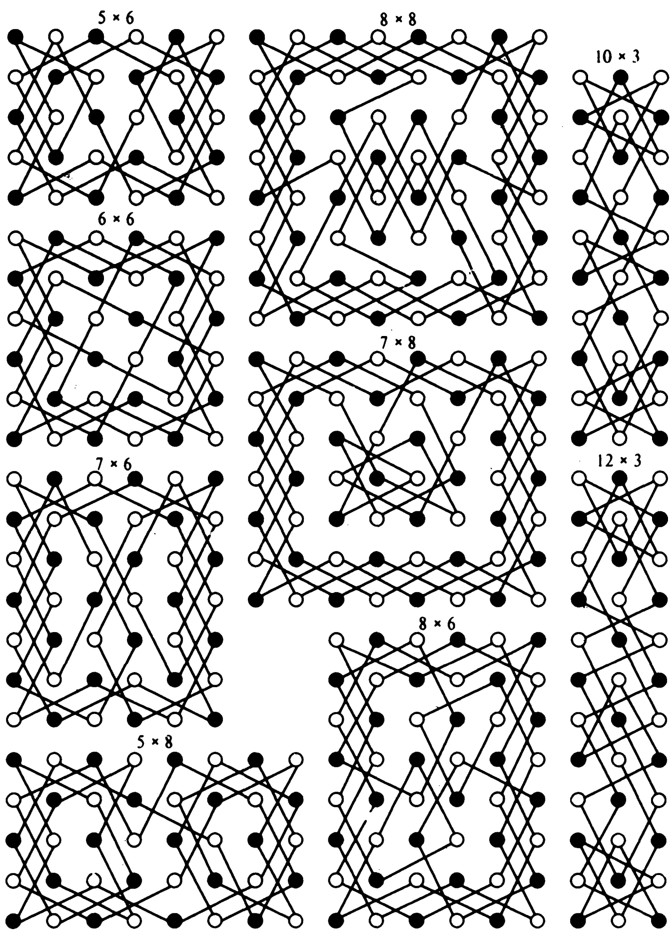

FIGURE 9

The nine Hamiltonian cycles that form the base of the inductive construction.

I like this illustration that induction can require many cases in order to get started because it shows students that they must be flexible when designing a proof by induction. I find this complete solution satisfying because the three conditions in the theorem are so easy to state, the impossible sizes are easy to understand, and while the inductive construction requires a bit of detail, the method remains totally elementary. Not everyone may wish to take class time to go through the complete solution, but students should be told that the full solution is entirely within their grasp.

Acknowledgement. The author is grateful to the referees for the detailed comments that greatly improved this article.

REFERENCES

1. C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam, 1973, p. 188.
2. G. Chartrand, *Introductory Graph Theory*, Dover Publications, Mineola, NY, 1985, pp. 133–135.
3. H. Dudeney, *Amusements in Mathematics*, Dover Publications, Mineola, NY, 1970, p. 103.
4. R. B. Eggleton and A. Eid, Knight's circuits and tours, *Ars Combinatoria* 17A (1984), 145–167.
5. R. Honsberger, Pósa's solution, *Mathematical Gems*, 1973, p. 145.
6. W. W. Rouse Ball and H. S. M. Coxeter, *Mathematical Recreations and Essays*, University of Toronto Press, Toronto, 1974, pp. 175–185.
7. R. J. Wilson and J. J. Watkins, *Graphs, an Introductory Approach*, John Wiley and Sons, Inc., New York, 1990, p. 145.

Steiner Minimal Trees on Chinese Checkerboards

F. K. HWANG
AT&T Bell Laboratories
Murray Hill, NJ 07974

D. Z. DU
Academia Sinica
Beijing, China 100010

1. Introduction For a given set P of points in the plane, $S(P)$, *a Steiner minimal tree* (SMT) is the shortest network spanning P. Recently, Chung, Gardner, and Graham [1] studied the SMT problem on an $n \times n$ square lattice that they called a *checkerboard*. They gave a general construction of heuristic SMTs on checkerboards and conjectured that the constructed trees are minimal for certain values of n. While the checkerboard is in the shape of a square, the Chinese checkerboard is in the shape of a Star of David triangulated into equilateral triangles of the same size (see FIGURE 1).

Formally, for $n \geq 2$ we define a latticed hexagon (triangle) to be a regular hexagon (equilateral triangle) divided into disjoint equilateral triangles by lines parallel to the sides of the hexagon (triangle). The hexagon (triangle) is said to be of order n if there are n lattice points on each side. Let $H_n(T_n)$ be the set of lattice points of the