Test 1 Solutions

1) Let R be the subring $\{a + b\sqrt{10} : a, b \in Z\}$ of the field of real numbers. Let $N : R \to Z$ be given by $N(a + b\sqrt{10}) = a^2 - 10b^2$.
a) Show that N is multiplicative, i.e., $N(uv) = N(u)N(v)$ for all $u, v \in R$, and show that $N(u) = 0$ if and only if $u = 0$.
b) Show that u is a unit in R if and only if $N(u) = \pm 1$.
c) Is 2 an irreducible element of R? What about $4 + \sqrt{10}$? Prove or disprove in each case.

Solution:
a) Let $u = a + b\sqrt{10}$ and $v = c + d\sqrt{10}$. Then

$$N(u)N(v) = (a^2 - 10b^2)(c^2 - 10d^2) = N(ac + 10bd + (bc + ad)\sqrt{10}) = N(uv).$$

b) Suppose that u is a unit. Then there is $v \in R$ with $uv = 1$ so by part a) $N(uv) = N(u)N(v) = 1$. Now $N(u) = \pm 1$. In the other direction, if $u = a + b\sqrt{10}$ and $a^2 - 10b^2 = \pm 1$, then $\pm(a - b\sqrt{10})$ is the inverse of u.
c) If $2 = uv$, then $4 = N(2) = N(u)N(v)$ so one of $|N(u)|$ or $|N(v)|$ is 1 or both are 2. In the first case, one of them is a unit, and in the second case, $a^2 - 10b^2 = \pm 2$. But such an equation is impossible with a, b integers (consider the equation mod 10). So 2 is irreducible. Now $N(4 + \sqrt{10}) = 6$ so if it can be factored, then one of the products has norm ± 2 which is impossible.

2) Let R be an integral domain with quotient field F. Let T be an integral domain such that $R \subset T \subset F$. Prove that F is (isomorphic to) the quotient field of T.

Solution: Let F_T be the quotient field of T. Define $f : F_T \to F$ by $f(x/y) = ad/bc$ where $x = a/b$, $a, b \in R$ and $y = c/d$, $c, d \in R$. It is easy (though tedious) to show that $f(x_1/y_1)f(x_2/y_2) = f(x_1x_2/y_1y_2)$ and that $f(x_1/y_1) + f(x_2/y_2) = f(x_1/y_1 + x_2/y_2)$. Thus f is a ring homomorphism. It is surjective since the preimage of a/b is x/y where $x = ac/c$ and $y = bd/d$ for some $c, d \in R$. Now if $f(x/y) = 0$ where $x = a/b$ and $y = c/d$, then $ad = 0$ and since R is an integral domain, and $d \neq 0$, we have $a = 0$. Thus $x = 0$ and $x/y = 0$. Consequently, $Ker(f) = 0$ and therefore F_T is isomorphic to F.

3) Let F be a field and $f, g \in F[x]$ with deg $g \geq 1$. Prove that there exist unique polynomials $f_0, f_1, \ldots, f_r \in F[x]$ such that deg $f_i <$ deg g for all i and

$$f = f_0 + f_1 g + f_2 g^2 + \cdots + f_r g^r.$$

Solution: There is an integer r such that $deg(g^r) \leq deg(f) < deg(g^{r+1})$ so by the division algorithm we get $f = f_r g^r + q_r$ where $deg(q_r) < deg(g^r)$. Now let q_r play the role of f. We then get an integer r_1 such that $deg(g^{r_1}) \leq deg(q_r) < deg(g^{r_1+1})$ and $r_1 < r$. Also $q_r = f_{r_1} g^{r_1} + q_{r_1}$. When this process terminates we have $f = \sum_{i=0}^{r} f_i g^i$ with $deg(f_i) < deg(g)$.
Now suppose that $f = \sum f_i g^i = \sum h_i g^i$. Then $\sum (f_i - h_i) g^i = 0$. Note that for $i < j$, $(f_i - h_i)g^i$ and $(f_j - h_j)g^j$ share no common power of x, so $(f_i - h_i)g^i = 0$ for all i. But F is a field, so $F[x]$ is a domain which means that $g^i = 0$ or $f_i - h_i$. Since $g \neq 0$, the latter holds and we're done with the proof of uniqueness.

4) Suppose that there are m red clubs $R_1, \ldots R_m$ and m blue clubs $B_1, \ldots, B_m$ in a town of n citizens. Assume that the clubs satisfy the following rules:
(a) $|R_i \cap B_i|$ is odd for every i;
(b) $|R_i \cap B_j|$ is even for every $i \neq j$.
Prove an upper bound for m in terms of n and give an example achieving it.

Solution: Let v_i be the incidence vector (over F_2) for club R_i and w_j be the incidence vector for club B_j. Suppose we have a linear combination $\sum c_i v_i = 0$. Dot product each side with w_j. The conditions of the theorem imply that we get $c_j = 0$. Thus the v_i's are linearly independent. Since they lie in a space of dimension n, there are at most n of them, so $m \leq n$. Letting $R_i = B_i = \{i\}$ for all i achieves this bound.