Generalizations of the Graham-Pollak Theorem

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Preliminaries

- Joint work with Sebastian Cioabă.
- All graphs will be finite. $A(G)$ will denote the adjacency matrix of a graph G.
- The terms *biclique* and *complete bipartite subgraph* will be used interchangeably.

Preliminaries

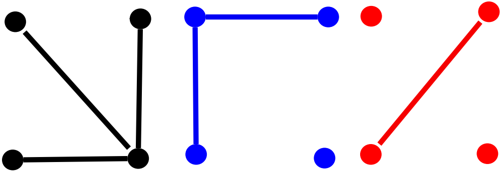
- First let us consider the problem of partitioning the edges of a graph by bicliques. Since each edge is a biclique, this can always be done. However, we want to use the fewest number of bicliques possible.

Definition

The **biclique partition number** of a graph G is the minimum number of bicliques necessary to partition the edges of a graph. We will denote it by $\mathrm{bp}(G)$.

- In general, this graph invariant is hard to compute.

- For a graph G, upper bounds on $\mathrm{bp}(G)$ come from constructions. We find bicliques whose edges partition the edge set of G.



- So for example, $\mathrm{bp}(K_4) \leq 3$.

Graham-Pollak Theorem

Theorem (Graham, Pollak 1972)

The edge set of a K_n cannot be partitioned into the edge disjoint union of less than $n-1$ complete bipartite subgraphs.

- $bp(K_n) \geq n-1$.
- This bound is tight, and there are many partitions of K_n into $n-1$ bicliques.
- For example, we can take $n-1$ "stars" (i.e. K_n is partitioned into $K_{1,n-1}, K_{1,n-2}, ..., K_{1,2}, K_{1,1}$).
- $\mathrm{bp}(K_n) = n-1$.

Proofs of the Graham-Pollak Theorem

- Linear algebra based proofs by Tverberg (1982), Witsenhausen (1980s), and G.W. Peck (1984).
- A polynomial space proof by Vishwanathan (2008)
- A counting proof by Vishwanathan (2010).

L-Coverings

In this talk we want to consider a generalization of the Graham-Pollak Theorem. Instead of requiring a partition of the edges of K_n, we require that the number of times each edge is covered comes from a specified list.

Definition

Let $L = \{l_1, ..., l_k\}$ where $0 < l_1 < ... < l_k$ are integers. An **biclique covering of Type L** of a graph G is a set of complete bipartite subgraphs of G that cover the edges of G such that the number of times each edge of G is covered is in L.

We will denote the minimum number of bicliques required for such a covering by $\mathrm{bp}_L(G)$.

L-coverings

- If $L = \{1\}$, then $\mathrm{bp}_L(G) = \mathrm{bp}(G)$.
- If $L = \mathbb{N}$, $\mathrm{bp}_L(K_n)$ is the biclique cover number: $\mathrm{bp}_{\mathbb{N}}(K_n) = \lceil \log_2 n \rceil$
- Exact results are known for very few lists L.
- For $L = \{1, 2, ..., t\}$, Alon gave bounds for $\mathrm{bp}_L(K_n)$ in 1997.
- Huang and Sudakov improved his lower bound recently.
  Next we will talk about some other lists.

General upper bounds

Given any list L, how can we find upper bounds for $\mathrm{bp}_L(K_n)$? We have the following recursive technique:

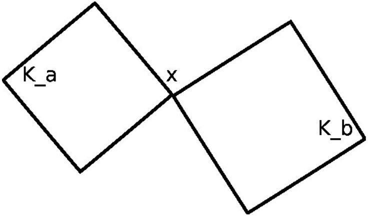
Proposition

For any list L, and any a and b

$$\mathrm{bp}_L(K_{a+b-1}) \leq \mathrm{bp}_L(K_a) + \mathrm{bp}_L(K_b).$$

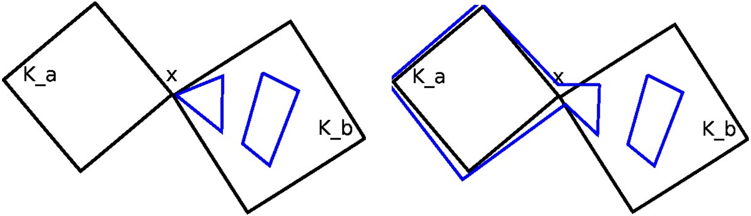
General upper bounds

- Let the vertex sets of K_a and K_b intersect on one vertex x.



- We will modify an optimal L-covering of K_a and of K_b
- Leave the $\mathrm{bp}_L(K_a)$ bicliques unchanged, modify the $\mathrm{bp}_L(K_b)$ bicliques in K_b into bicliques in K_{a+b-1}.
- If a biclique contains x, say $x \in U$, then replace it by $(V(K_a) \cup U, V)$.

General upper bounds



Edges that are completely inside K_a or K_b are covered the number of times that they were before. Edges pq with $p \in A \setminus \{x\}$ and $q \in B \setminus \{x\}$ are covered the same number of times as the edge xq which is in K_b. Thus all edges are L-covered.

Odd cover problem

Suppose now we ask the question, how many bicliques are necessary to cover K_n such that each edge is covered an odd number of times?

- So we are asking for $\mathrm{bp}_L(K_n)$ where $L = \{1, 3, 5, 7, \ldots\}$.
- This question was first asked by Babai and Frankl in 1992.
- It is called the odd-cover problem.

Proposition (Cioabă and MT, 2012)

If $L = \{1, 3\}$, then

$$\frac{n-1}{2} \leq \mathrm{bp}_L(K_n) \leq \frac{4n}{7} + 2.$$

Odd cover problem

Proof:

- For the lower bound, let $\{B_i(U_i, W_i)\}_{i=1}^d$ be bicliques that cover K_n such that each edge is covered either 1 or 3 times.
- We want to write $A(K_n)$ as a linear combination of matrices.

$$A(K_n) = \sum_{i=1}^{d} A(B_i) - 2 \sum_{1 \leq i < j < k \leq d} A(B_i \cap B_j \cap B_k).$$

Odd cover problem

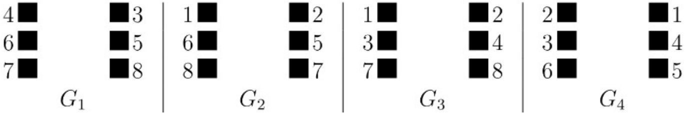
Reducing over $\mathbb{F}_2$, we have

$$A(K_n) \equiv \sum_{i=1}^{d} A(B_i) \pmod 2$$

- We use subadditivity of rank to complete the proof.
- Since $A(K_n)$ has rank at least $n-1$ over $\mathbb{F}_2$, and each $A(B_i)$ has rank 2, we have $2d \geq \mathrm{rank}\left(\sum_{i=1}^{d} A(B_i)\right) \geq n-1$.

Odd cover problem

For the upper bound, $\mathrm{bp}_L(K_8) = 4$.



Now we use the recursion from before and induction.

$$\mathrm{bp}_L(K_n) \leq \mathrm{bp}_L(K_{n-7}) + \mathrm{bp}_L(K_8).$$

We note that the same lower bound holds for $L = \{1, 3, 5, 7, ...\}$ with the same proof technique.

Even cover problem

We might ask the same question for even instead of odd.

- For $L = \{2, 4, 6, ...\}$, what is $\mathrm{bp}_L(K_n)$?
- Given the answer to the previous problem, we might expect the answer to be linear.
- Surprisingly, it is not.

Proposition

For $L = \{2, 4, 6, ...\}$,

$$\mathrm{bp}_L(K_n) = \lceil \log_2 n \rceil + 1.$$

$L = \{\lambda\}$

- Now let's consider the list $L = \{\lambda\}$ for a fixed λ.
- $\mathrm{bp}_L(K_n) = \mathrm{bp}(\lambda K_n)$ where λK_n is the complete multigraph.
- The lower bound is $\mathrm{bp}_{\{\lambda\}}(K_n) \geq n-1$. The proof is the same as for the Graham-Pollak Theorem.
- de Caen conjectured in 1993 that for any λ, for every n larger than some N_λ, $\mathrm{bp}_{\{\lambda\}}(K_n) = n-1$.
- This conjecture is true for $\lambda \leq 18$.
- Perhaps we can use the recursion to show $\mathrm{bp}_{\{\lambda\}}(K_n) \leq n + c_\lambda$ for n large enough.

Graham-Pollak for hypergraphs

- We can also generalize the Graham-Pollak Theorem to hypergraphs.
- We ask, how many complete r-partite r-uniform hypergraphs are necessary to partition the edge set of the complete r-uniform hypergraph on n vertices.
- We denote this quantity by $f_r(n)$.

Graham-Pollak for hypergraphs

- $f_2(n) = \mathrm{bp}(K_n) = n - 1$.
- $f_3(n) = n - 2$.
- $f_r(n) = \Theta(n^{\lceil r/2 \rceil})$.
- In general, this problem seems very hard.

Graham-Pollak for hypergraphs

Theorem - Cioabă, Kündgen, Verstraëte (2009)

$$\frac{2\binom{n-1}{k}}{\binom{2k}{k}} \leq f_{2k}(n)$$

and

$$f_{2k}(n) \leq f_{2k+1}(n+1) \leq \binom{n-k}{k}.$$

This improved a result of Alon.

Graham-Pollak for hypergraphs

Theorem - Cioabă and MT (2012)

$$f_{2k}(2k+2) = \lceil \frac{2k^2+5k+3}{4} \rceil$$

and

$$f_{2k+1}(2k+3) = \lceil \frac{2k^2+5k+3}{4} \rceil.$$

This can be used to improve the general upper bound by a lower order term.

Open Problems

- For any fixed λ, can we prove $\mathrm{bp}_{\{\lambda\}}(K_n) \leq n + c_\lambda$?
- For fixed L, is $\mathrm{bp}_L(K_n) = \Theta(n^{1/k})$ for some fixed k?
- What is $f_4(n)$?