# Math 301 Homework

Mary Radcliffe

Due 3 November 2017

Complete the following problems. Fully justify each response.

1. (a) Let $G$ be a graph of order $n \geq 4$, and having $m \geq \frac{n+n\sqrt{4n-3}}{4} + 1$. Prove that $G$ contains a 4-cycle.

   (b) Let $G$ be a graph of order $n \geq 4$ and having $m \geq \frac{n}{2}\sqrt{n-1}+1$ edges. Prove that $G$ contains either a 3-cycle or a 4-cycle.

2. Let $G = (V, E)$ be a graph with average degree $D$. Show that there exists a subgraph $H$ in $G$ with $\delta(H) \geq \frac{D}{2}$, that is, the minimum degree in $H$ is at least $\frac{D}{2}$.

3. Let $T$ be a tree on $k$ vertices. Prove that $\frac{n(k-2)}{2} \leq ex(n, T) \leq nk$ for any $n \in \mathbb{N}$ having $n$ divisible by $k - 1$. Prove that there exists at least some trees for which the lower bound is in fact equal to $ex(n, T)$.

   NOTE: We do not know the extremal numbers for trees, in general. It has been conjectured (Erdős-Sós Conjecture) that the lower bound above is in fact equal to the extremal number for all trees, but the conjecture has been open since 1963.

4. For $0 < s \leq t \leq n$, define $z(n, s, t)$ to be the maximum number of edges in a bipartite graph $G$, with partite sets $V_1, V_2$ such that $|V_1| = |V_2| = n$ and $K_{s,t} \not\subseteq G$. Prove that $2ex(n, K_{s,t}) \leq z(n, s, t) \leq ex(2n, K_{s,t})$.