Math 301: Homework 2

Mary Radcliffe

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Complete the following problems. Fully justify each response.

1. For $a, b \in \mathbb{Z}$, let $L(a, b)$ denote the set of lattice paths from $(0, 0)$ to (a, b). Fix $n \in \mathbb{Z}$, and let $1 \leq k \leq n-1$. Construct an injective function

   $$F : L(k-1, n-k+1) \times L(k+1, n-k-1) \to L(k, n-k) \times L(k, n-k).$$

   Use this to prove that for all $k \leq n-1$,

   $$\binom{n}{k-1}\binom{n}{k+1} \leq \binom{n}{k}^2.$$

   (This property is called *log-concavity*. That is, a sequence $a_0, a_1, \ldots, a_n$ is called log-concave if for all $1 \leq k \leq n-1$, we have $a_{k-1}a_{k+1} \leq a_k a_k$. Hence, here we are proving that the sequence $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ is log-concave.)

2. Prove that $\displaystyle\sum_{a_1+a_2+a_3=n} \binom{n}{a_1, a_2, a_3} (-1)_2^a = 1.$

3. Suppose a class contains $2n$ students, and they have projects to work on in pairs.
   (a) How many different ways can the professor pair the students up?
   (b) After the first project, the professor wishes to create new pairings, so that nobody will be working with the same partner a second time. How many ways can the professor create this second set of pairings?