Midterm 3 Review Solutions by CC

Problem 1

Set up (but do not evaluate) the iterated integral to represent each of the following.

(a) The volume of the solid enclosed by the parabaloid $z = 2x^2 + 2y^2 - 1$ and the planes $z = 1$, $x = \pm\frac{1}{2}$.

Clearly, the range for the x coordinate is given by $\left[-\frac{1}{2}, \frac{1}{2}\right]$ as it must lie between the the planes. The z coordinate is beteween 1 and $f(x, y) = 2x^2 + 2y^2 - 1$. We observe that for $|x| < \frac{1}{2}$and y small, $f(x, y) < 1$ so that $f(x, y)$ should be the lower limit, and 1 the upper limit.

Finally, we find the boundary of the region for the y coordinate. This occurs when the two surfaces intersect, i.e. $1 = 2x^2 + 2y^2 - 1 \Rightarrow y = \pm\sqrt{1 - x^2}$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\int_{2x^2+2y^2-1}^{1} 1dz \right) dy \right) dx$$

(b) The volume of the solid beneath the surface $z = xy$ and above the triangle having coordinates $(1, 0)$, $(0, 1)$, and $(2, 2)$.

The z coordinate will range from 0 to xy. The range for the x and y coordinates in the region bounding the given triangle in the xy plane. Note that the x coordinate ranges from 0 to 2, and the y coordinate can be bounded between $1 - x$ and $1 + \frac{x}{2}$ for $0 \le x \le 1$, and between $2x - 2$ and $1 + \frac{x}{2}$ for $1 \le x \le 2$. Therefore the volume can be expressed as

$$\int_0^1 \int_{1-x}^{1+\frac{x}{2}} \int_0^{xy} 1dzdydx + \int_1^2 \int_{2x-2}^{1+\frac{x}{2}} \int_0^{xy} 1dxdydx$$

(c) The average value of the function $z = (x + y)^2$ over the rectangle $[0, 2] \times [0, 2]$.

We want to integrate the function over the given rectangle, then average it. This is given by

$$\frac{\int_0^2 \int_0^2 (x+y)^2 \, dydx}{\int_0^2 \int_0^2 1dydx} = \frac{1}{4} \int_0^2 \int_0^2 (x+y)^2 \, dydx$$

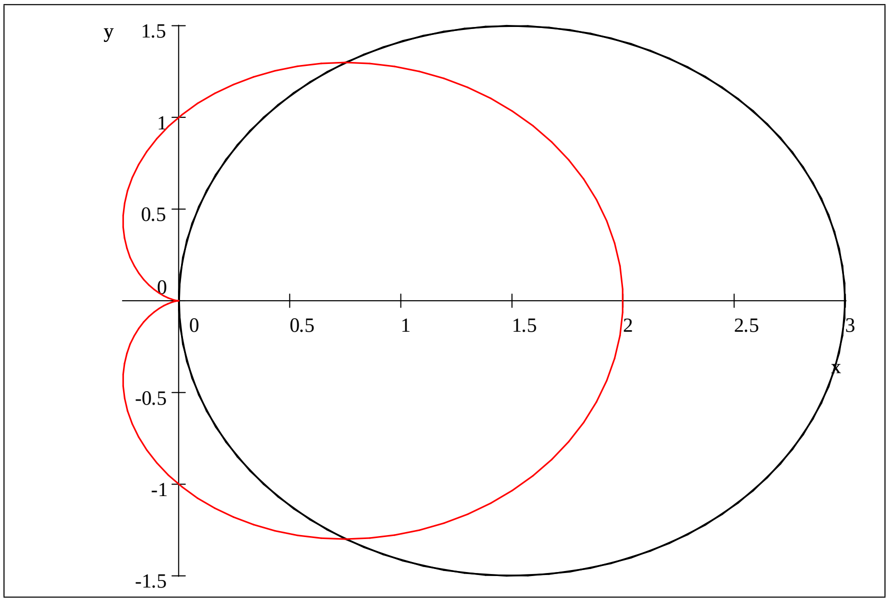
(d) The area of the region outside $r = 3\cos\theta$ and inside $r = 1 + \cos\theta$.

Since the curves are given in polar coordinates, we will stay in that system and express the integral in that manner. First, note that the two curves intersect when $3\cos\theta = 1 + \cos\theta$, i.e. when $2\cos\theta = 1$ or $\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ and also at $-\frac{\pi}{3}$. By symmetry, it suffices to consider the region between $\frac{\pi}{3}$ and π, and then multiply it by a factor of 2.

Therefore, the area between them will be given by

$$2\left\{ \int_{\frac{\pi}{3}}^{\pi} \left(\int_0^{1+\cos\theta} rdr \right) d\theta - \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\int_0^{3\cos\theta} rdr \right) d\theta \right\}$$

Below, $r = 1 + \cos\theta$ is pictured in red and $r = 3\cos\theta$ in black.



(e) The volume of the solid between the cone $z^2 = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 4$.

To ease notation, it makes sense to solve this problem in cylindrical or spherical coordinates rather than rectangular. We do so in cylindrical coordinates. Rewriting, the cone is then given by

$$z = \pm r$$

and sphere by

$$z = \pm\sqrt{4 - r^2}.$$

There will be two halves to this volume, symmetric about the xy plane. We thus (equivalently) write the expression for twice the volume of the top half.

The z coordinate will range from r^2 to $\sqrt{4 - r^2}$ (above the cone, beneath the sphere). The radial coordinate can range from 0 (at the origin) to at most the outer boundary when the sphere and cone meet, i.e. when

$$r_0^2 = 4 - r_0^2 \Rightarrow r_0 = \sqrt{2}.$$

Finally, the θ coordinate is unrestricted since we have symmetry. Thus, the integral can be written as

$$\int_0^{2\pi} \left(\int_0^{\sqrt{2}} \left(\int_r^{\sqrt{4-r^2}} r dz \right) dr \right) d\theta$$

(f) The area of the portion of the hyperbolic paraboloid $z = x^2 - y^2$ between the cylinders $x^2 + y^2 = 4$ and $x^2 + z^2 = 9$.

We want to compute the surface area in the annulus between $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. For the given surface $z = x^2 - y^2$, we have

$$\frac{\partial z}{\partial x} = 2x, \ \frac{\partial z}{\partial y} = -2y$$

so that

$$\begin{aligned} A &= \int\int_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA \\ &= \int\int_D \sqrt{1 + x^2 + y^2} dA \end{aligned} \tag{1}$$

We could write out the limits of integration in rectangular coordinates, but it's cleaner to switch to polar. Then the boundaries are given by $r = 2$ and $r = 3$ and (1) yields

$$A = \int_0^{2\pi} \int_2^3 r\sqrt{1 + r^2} dr d\theta$$

(g) The area of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 2y$.

We will break this problem into three parts, two from the top part of the sphere and one from the bottom part. In the plane $y > 0$, the relevant area is just that of the top half of the sphere that lies above $2y$, and in the plane $y < 0$ we must account for the entire top and some of the bottom. We will write

$$\begin{aligned} z_{top} &= +\sqrt{4 - x^2 - y^2} \\ z_{bot} &= -\sqrt{4 - x^2 - y^2} \end{aligned}$$

noting

$$\begin{aligned} \frac{\partial z_{top}}{\partial x} &= -\frac{x}{\sqrt{4 - x^2 - y^2}}, \ \frac{\partial z_{top}}{\partial y} = -\frac{y}{\sqrt{4 - x^2 - y^2}} \\ \frac{\partial z_{bot}}{\partial x} &= \frac{x}{\sqrt{4 - x^2 - y^2}}, \ \frac{\partial z_{bot}}{\partial y} = \frac{y}{\sqrt{4 - x^2 - y^2}} \end{aligned}$$

The intersection of the plane and the sphere occurs when

$$\begin{aligned} 2y &= \sqrt{4 - x^2 - y^2} \\ 5y^2 - 4 &= -x^2 \\ y^2 &= \frac{4 - x^2}{5} \\ y &= \sqrt{\frac{4 - x^2}{5}} \end{aligned}$$

Then the contribution from the $y > 0$ half above the xy plane is given by

$$\begin{aligned}
A_1 &= \int\int_{D_1} \sqrt{1 + \left(\frac{\partial z_{top}}{\partial x}\right)^2 + \left(\frac{\partial z_{top}}{\partial y}\right)^2} dA \\
&= \int\int_{D_1} \sqrt{1 + \frac{x^2+y^2}{4-x^2-y^2}} dA \\
&= \int_{-2}^{2}\int_{0}^{\sqrt{\frac{4-x^2}{5}}} \sqrt{1 + \frac{x^2+y^2}{4-x^2-y^2}} dydx \\
&= \int_{0}^{\pi}\int_{0}^{\frac{2}{\sqrt{1+4\sin^2\theta}}} \sqrt{1 + \frac{r^2}{4-r^2}} rdrd\theta
\end{aligned}$$

and the contribution from $y < 0$ above the xy plane by

$$\begin{aligned}
A_2 &= \int\int_{D_2} \sqrt{1 + \left(\frac{\partial z_{top}}{\partial x}\right)^2 + \left(\frac{\partial z_{top}}{\partial y}\right)^2} dA \\
&= \int\int_{D_2} \sqrt{1 + \frac{x^2+y^2}{4-x^2-y^2}} dA \\
&= \int_{\pi}^{2\pi}\int_{0}^{2} \sqrt{1 + \frac{r^2}{4-r^2}} rdrd\theta,
\end{aligned}$$

since the domain is the whole semicircle. Finally, the part below the xy-plane but above $z = 2y$ is given (by symmetry) as A_1, and the result is

$$A = 2A_1 + A_2.$$

(h) The volume bounded by the cone $z^2 = x^2 + y^2$, the cylinder $x^2 + y^2 = 16$, and the planes $z = -3$ and $z = 4$.

We will solve this problem in cylindrical coordinates for convenience. First, we rewrite the equations of the cone and cylinder as

$$z = \pm r, \; r = 4,$$

respectively. We will calculate the parts $0 < z < 4$ and $-3 < z < 0$ separately. The former is

$$\begin{aligned}
&\int\int\int_V rdrdzd\theta \\
&= \int_0^{2\pi}\int_0^4\int_z^4 rdrdzd\theta
\end{aligned}$$

and the latter as

$$\int_0^{2\pi}\int_{-3}^{0}\int_{-z}^{4} rdrdzd\theta$$

Problem 3

Calculate the volume of the solid enclosed by the surface $z = 1 + x^2 y e^y$ and the planes $z = 0$, $x = \pm 1$, $y = 0$, and $y = 1$.

The domain is given by $-1 < x < 1$ and $0 < y < 1$, so the volume underneath the graph can be expressed as

$$\begin{aligned}
&\int_{-1}^{1} \int_{0}^{1} \left(1 + x^2 y e^y\right) dy dx \\
&= \int_{-1}^{1} \left(y + x^2 y e^y - x^2 e^y\right)_{y=0}^{y=1} dx \\
&= \int_{-1}^{1} \left(1 + x^2 e - x^2 e + x^2\right) dx \\
&= \left(x + \frac{x^3}{3}\right)_{x=-1}^{x=1} = 2 + \frac{2}{3} = \frac{8}{3}
\end{aligned}$$

Problem 5

Suppose you have a solid ball of radius 2. Find the average distance between a point in the ball and the center of the ball.

We are asked to find the average distance between a point in a ball and its center. Center the ball at the origin for simplicity, and use spherical coordinates. The distance from an arbitrary point (ρ, θ, ϕ) to the center $(0, 0, 0)$ is simply ρ. The average distance is then ρ integrated over the sphere, normalized by the total volume of the sphere. As we know the volume of a sphere of radius R is $\frac{4}{3}\pi R^3$, we then have

$$\begin{aligned}
d &= \left(\frac{4}{3}\pi \left(2\right)^3\right)^{-1} \int\int\int_S \rho\rho^2 \sin\phi d\rho d\phi d\theta \\
&= \frac{3}{32\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^{2} \rho^3 \sin\phi d\rho d\phi d\theta = \frac{3}{32\pi} \left(4\pi\right) \left(\frac{\rho^4}{4}\right)_{\rho=0}^{\rho=2} \\
&= \frac{3}{8} \left(\frac{16}{4}\right) = \frac{3}{2}
\end{aligned}$$

Problem 7

Calculate $\int\int_D \left(x^2 + y^2\right)^{\frac{3}{2}} dA$ where D is the region in the first quadrant of the xy plane bounded by $y = 0$, $y = \sqrt{3}x$, and $x^2 + y^2 = 9$.

It makes sense to transform to polar coordinates to evaluate this problem due to the structure of the integrand. We first rewrite the bounds on the region given in terms of polar coordinates:

$$\begin{aligned}
y = 0 &\Rightarrow \theta = 0 \\
y = \sqrt{3}x &\Rightarrow \theta = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3} \\
x^2 + y^2 = 9 &\Rightarrow r = 3.
\end{aligned}$$

This is a sector of a circle, in particular, the section from $\theta = 0$ to $\theta = \frac{\pi}{3}$ of the circle of radius 3. We now perform the integration:

$$\begin{aligned}\int\int_D \left(x^2+y^2\right)^{\frac{3}{2}} dA &= \int_0^{\frac{\pi}{3}} \int_0^3 r^3 r dr d\theta \\ &= \int_0^{\frac{\pi}{3}} \left(\frac{r^5}{5}\right)_{r=0}^{r=3} d\theta \\ &= \frac{243}{5}\left(\frac{\pi}{3}\right) = \frac{81}{5}\pi.\end{aligned}$$

Problem 9

Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $F = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$ and C is obtained by following the parabola $y = 1 + x^2$ from the point $(-1, 2)$ to the point $(1, 2)$.

Method 1. Observe that F is a conservative field, for

$$\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = -\frac{xy}{\sqrt{x^2+y^2}} = \frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

and in fact, for

$$f := \sqrt{x^2+y^2},$$

we have $\nabla \vec{f} = \vec{F}$. Thus, by the fundamental theorem for line integrals,

$$\int_C \vec{F} \cdot d\vec{r} = f|_{(-1,2)}^{(1,2)} = \sqrt{1^2+2^2} - \sqrt{(-1)^2 + 2^2} = 0.$$

Method 2. We can also solve this problem by straight computation, parametrizing the curve by

$$x(t) = t-1, \ y(t) = 1 + (t-1)^2 = t^2 - 2t + 2$$

for $0 \le t \le 2$. Then we have

$$\vec{F}(\vec{r}(t)) = \left(\frac{t-1}{\sqrt{1+2(t-1)^2}}, \frac{t^2-2t+2}{\sqrt{1+2(t-1)^2}}\right)$$

and

$$dx = 1dt, \ dy = (t-2)\,dt$$

so that

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^2 \left(\left(\frac{t-1}{\sqrt{1+2(t-1)^2}}\right)(1) + \left(\frac{t^2-2t+2}{\sqrt{1+2(t-1)^2}}\right)(2t-2)\right) dt \\ &= 0,\end{aligned}$$

matching our answer from method 1.

Problem 11

Determine if each of the following vector fields is conservative. If so, find a function f such that $\vec{F} = \nabla f$.

(a) $\vec{F} = (2x+y)\,\hat{\imath} + \left(x+3y^2\right)\hat{\jmath}$

Set $P = 2x+y$, $Q = x+3y^2$. We then note

$$\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x},$$

implying the vector field is conservative. To compute f, we note that

$$f = \int P dx = \int (2x+y)\,dx = x^2 + yx + g(y)$$

where g is a function of y but not x. We differentiate f to find g:

$$f_y = x + g'(y) = Q = x + 3y^2$$

and conclude $g(y) = y^3 + K$ for a constant K. Therefore

$$f = x^2 + yx + y^3 + K,\ \ K \in \mathbb{R}.$$

To check, we can differentiate and verify

$$\begin{aligned} f_x &= 2x + y = P \\ f_y &= x + 3y^2 = Q \end{aligned}$$

(b) $\vec{F} = (2x+2y)\,\hat{\imath} + \left(x+3y^2\right)\hat{\jmath}$

Set $P = 2x+2y$, $Q = x+3y^2$.We note

$$\frac{\partial P}{\partial y} = 2$$

but that

$$\frac{\partial Q}{\partial x} = 1.$$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, we conclude the vector field $\vec{F}$ is not conservative.

(c) $\vec{F} = (x\cos y + 2)\,\hat{\imath} + (-x\sin y + 2x)\,\hat{\jmath}$

Set $P = x\cos y + 2$, $Q = -x\sin y + 2x$. We have

$$\frac{\partial P}{\partial y} = -x\sin y,$$

but

$$\frac{\partial Q}{\partial x} = -\sin y + 2.$$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, we conclude the vector field $\vec{F}$ is not conservative.

(d) $\vec{F} = \left(2x\cos\left(x+y^2\right) - x^2\sin\left(x+y^2\right)\right)\hat{\imath} - 2x^2y\sin\left(x+y^2\right)\hat{\jmath}$

Set $P = 2x\cos\left(x+y^2\right) - x^2\sin\left(x+y^2\right)$, $Q = -2x^2y\sin\left(x+y^2\right)$. Then we have

$$\begin{aligned}
\frac{\partial P}{\partial y} &= -4xy\sin\left(x+y^2\right) - 2yx^2\sin\left(x+y^2\right) \\
\frac{\partial Q}{\partial x} &= -4xy\sin\left(x+y^2\right) - 2x^2y\cos\left(x+y^2\right),
\end{aligned}$$

which are equal, implying that the vector field is conservative. To compute f, we note

$$\begin{aligned}
f &= \int P dx = \int \left(2x\cos\left(x+y^2\right) - x^2\sin\left(x+y^2\right)\right) dx \\
&= x^2\cos\left(x+y^2\right) + g(y)
\end{aligned}$$

where g is a function of y but not x. To find g we differentiate with respect to y:

$$f_y = -2x^2y\cos\left(x+y^2\right) + g'(y) = Q = -2x^2y\cos\left(x+y^2\right)$$

and conclude $g(y) = K$ for some constant K. Therefore

$$f = x^2\cos\left(x+y^2\right) + K$$

for $K \in \mathbb{R}$. To check, we differentiate and verify

$$\begin{aligned}
f_x &= 2x\cos\left(x+y^2\right) - x^2\sin\left(x+y^2\right) = P \\
f_y &= -2yx^2\sin\left(x+y^2\right) = Q
\end{aligned}$$

Problem 13

Suppose $\vec{F}$ is a conservative vector field and C is a closed curve. Prove that $\int_C \vec{F}\cdot d\vec{r} = 0$. You can use Green's Theorem.

Note: There are a number of other conditions required for the application of Green's Theorem. We are assuming that these other conditions hold.

Write $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$. Since $\vec{F}$ is a conservative vector field, we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Furthermore, by Green's Theorem, for a closed curve C:

$$\int_C \vec{F}\cdot d\vec{r} = \int\int_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dA = \int\int_D 0 dA = 0$$

where D is the region enclosed by the curve C.