Math 228: Homework 2

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Complete the following problems. Fully justify each response. You need only turn in those problems marked with a (*).

1. (*) Select 38 positive integers, all less than 1000. Prove that there must be two of them whose difference is less than 27.
2. (*) Prove the identity
$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0.$$
3. You are given a set S of $n+1$ distinct integers from $[2n]$.
   (a) Prove that there must exist $a, b \in S$ having no common divisor.
   (b) Is it true that there must exist $a, b \in S$ having $a = 2b$? If so, prove it. If not, provide a counterexample.
   (c) Is it true that there must exist $a, b \in S$ having $a = kb$ for some integer k? If so, prove it. If not, provide a counterexample.
4. (*) Prove that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.
5. How many ways can we permute the characters $\{1, 1, 2, 2, 3, 4, 5, 5\}$ so that no two identical digits are in consecutive positions?
6. (*) Given a positive integer n, we define $\phi(n)$ to be the number of integers in $[n]$ that share no divisors with n. For example, if $n = 6$, $\phi(n) = 2$, since the integers in $\{1, 2, 3, 4, 5, 6\}$ that share no divisor with 6 are 1 and 5. This is called Euler's totient function, or, more colloquially, Euler's ϕ-function.
   (a) Calculate $\phi(n)$ for $1 \leq n \leq 10$.
   (b) Prove that if p is prime and $k \geq 1$, then $\phi(p^k) = p^k \left(1 - \frac{1}{p}\right)$.
   (c) Use Inclusion-Exclusion to prove that if p and q are both prime, with $p \neq q$, and $k, j \geq 1$, then $\phi(p^k q^j) = p^k q^j \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)$.
   (d) Generalize part (c) to come up with a formula for $\phi(n)$ for any n. Prove that your formula is correct.