Math 228: Embedding graphs in surfaces

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1 Introduction

As we saw in the text, a planar graph is one that can be embedded into the plane (or sphere) in such a way that no edges cross each other. For example, the graph G shown in Figure 1 is planar, and is shown together with a plane embedding.

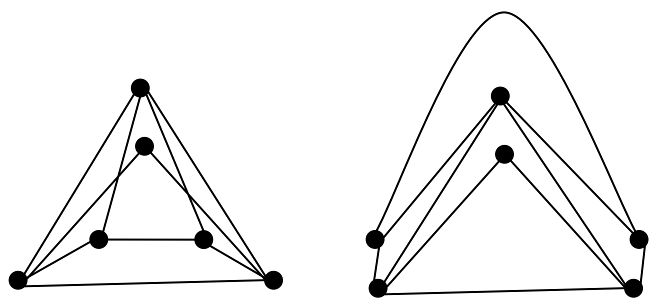

Figure 1: On the left is a planar graph G, drawn with edges crossing. The plane embedding of G on the right is obtained by moving the two vertices inside the large triangle out, and moving the edge between them so that it goes around the exterior of the triangle. A graph G that is planar together with a plane drawing of G is called a *plane graph*.

Of course, there are some graphs that cannot be drawn in the plane without their edges crossing. We shall see in future lectures/notes a characterization of such graphs known as Kuratowski's Theorem. For now, let us take as true (you can prove it, if you like) that the complete bipartite graph $K_{3,3}$ (see Figure 2) cannot be drawn in the plane without edges crossing.

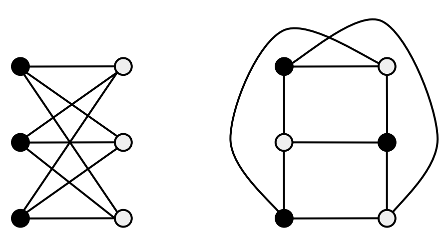

Figure 2: Two drawings of the complete bipartite graph $K_{3,3}$. On the left, we have the 'standard' drawing of a complete bipartite graph $K_{k,\ell}$, having k black vertices, ℓ white vertices, and all possible edges in between. On the right, we have a drawing in which we have made the fewest possible edge crossings.

Notice that if we had a way to bring one of the round edges in the drawing of $K_{3,3}$ on the right "around the back," we might be able to draw the graph without the edges crossing. As in class, we note that this cannot be done in the sphere, as a drawing in the sphere (see Figure 3) will necessarily have the same problem as in the plane: one of the curved edges cuts the space into two parts, and the other curved edge must connect one vertex from each of those two parts.

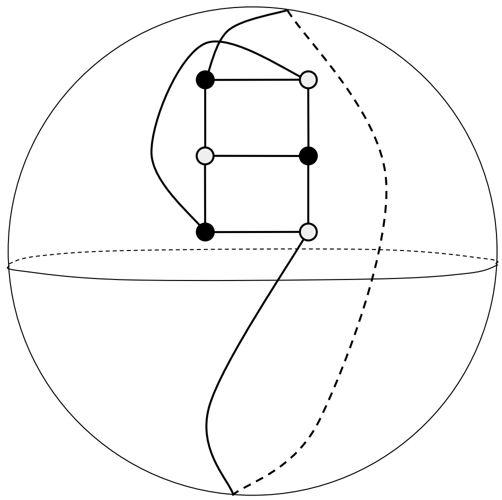

Figure 3: An attempt to draw $K_{3,3}$ on the sphere. Dashed lines indicate edges that go along the back of the sphere. Notice that the long edge being sent behind the sphere still cuts the space into two halves, and the remaining edge is forced then to cross the line in order to connect the two vertices from the two halves.

Hence, we must turn to other surfaces in order to create a new direction for our edges to go.

2 Toroidal Embeddings

The first and simplest nonspherical surface is the torus, obtained from the sphere by poking a hole through the center, as shown in Figure 4.

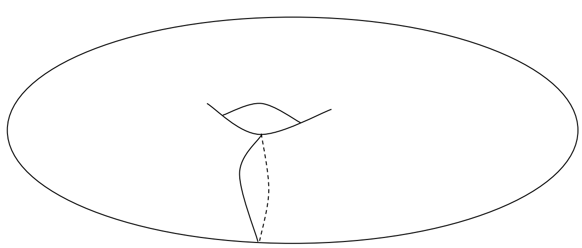

Figure 4: A basic torus. Shaped like the surface of a donut.

The torus is essentially the shape of the surface on a donut; you can imagine it as the powdered sugar layer of a donut. Notice that in the torus, we have two different ways to draw circles: we can draw them through the hole, or around the top of the shape. This will give us more freedom in drawing our graphs on the torus. Indeed, by using the two different directions, we can now successfully embed the $K_{3,3}$ into the torus as we desired (see Figure 5).

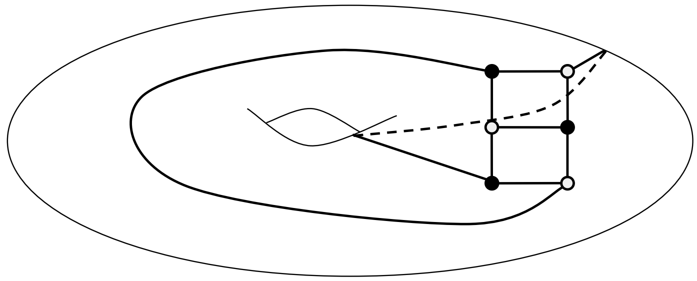

Figure 5: $K_{3,3}$ embedded on the torus. Notice that for the lines that would have crossed, we use the hole in the surface to give us a new direction to take one line, so that we need not cross them at all.

We now consider the question of vertices, edges, and faces drawn of $K_{3,3}$ drawn on the torus. Recall that we have the following theorem for plane graphs:

Theorem 1 (Euler's Formula)**.** *Let G be a plane graph. Let v denote the number of vertices of G, e the number of edges of G, and f the number of faces in the associated plane drawing of G. Then $v-e+f=2$.*

We proved this theorem in class, and a proof is also found in your text in Section 12.1. We would like to consider whether a similar formula holds for the torus. In order to do so, we'd like a more 2-dimensional way of thinking about the torus, as it can be difficult to see and understand the regions in a 3-dimensional drawing.

To this end, we shall build what's known as the "flat" torus: a way to think about the torus 2-dimensionally. This understanding of the torus is illustrated and explained in Figure 6.

Using the flat torus for our embedding, we can redraw $K_{3,3}$ as we did on the 3-dimensional torus, noting that when we have edges that go out on the left, they come back in on the right, and likewise for the top and bottom. We can then trace the regions of $K_{3,3}$ by tracing around their boundaries. These regions are indicated with different colors in Figure 7. To avoid confusion, we use black edges for the sides of the flat torus, rather than colored edges as in Figure 6, with different kinds of arrows to indicate the different identifications. Notice that we obtain 3 faces. Taking the data for $K_{3,3}$, we have 6 vertices, 9 edges, and 3 faces, and hence $v-e+f=0$, rather than 2 as before.

If we consider embedding other graphs in the torus (some examples are shown in Figure 8), we find that this pattern repeats itself: we always obtain $v-e+f=0$ (try it on the graphs in Figure 8!). Indeed, as with Euler's Formula (and proved in an identical way), it will always be the case in the torus that this equality holds.

More generally, we can consider embedding graphs in other surfaces. There will be some graphs that cannot embed either in the plane or the torus; for these we can add more holes to give ourselves more directions to draw edges (see Figure 9). In general, we will obtain the following theorem.

Theorem 2 (Euler's Formula, generalized)**.** *Let G be a connected graph associated with a drawing of G in a torus having g holes, for which every face in the drawing is 2-dimensional. Then $v-e+f=2-2g$.*

The letter g here is used for the word *genus*. In general, the genus of a torus is the number of holes it has. In fact, the word genus is used for graphs also; the *genus* of a graph is the minimum number of holes required to embed the graph in a torus. So the genus of $K_{3,3}$ is 1, but the genus of, say K_4 is 0 since we don't require any holes to embed K_4 (it is planar).

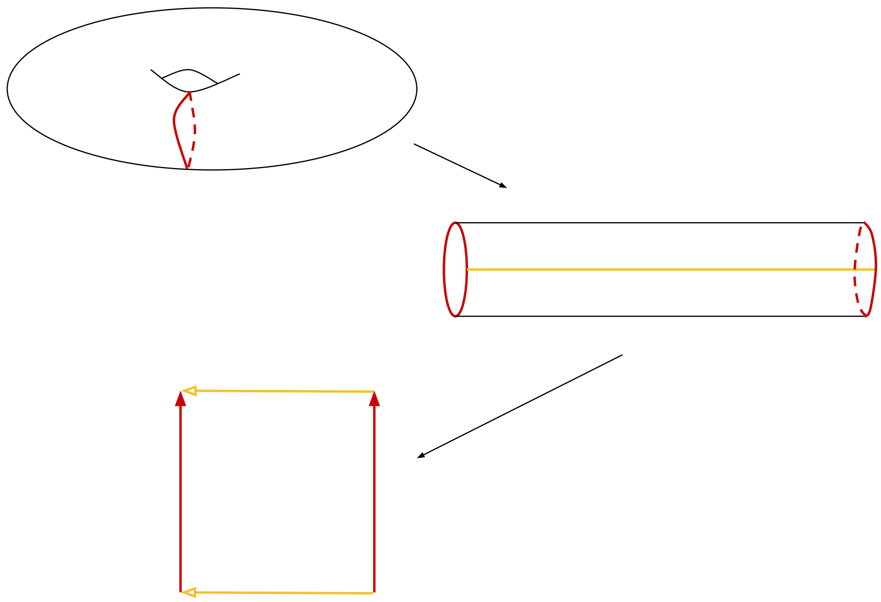

Figure 6: The construction of the flat torus. Beginning with the standard torus, imagine cutting along the red circle and "unfolding" the resulting cylinder. We keep the ends both red to indicate that in fact those are the same circles, we have just unfolded them. We then cut again along the yellow circle (note: this is a circle, since the two ends are on the red and are thus actually the same point) and unfold again, leaving us with a square. We draw arrows on the top and bottom (and left and right) to indicate that these are in fact the same edge. Drawing a line on the flat torus will operate as in Pac-man: if you go out the left, you will come back out in the exact same spot on the right, and likewise for top and bottom. In this way we can view graphs embedded in the torus on a 2-dimensional space (the flat torus) and thus manipulate and understand them more easily.

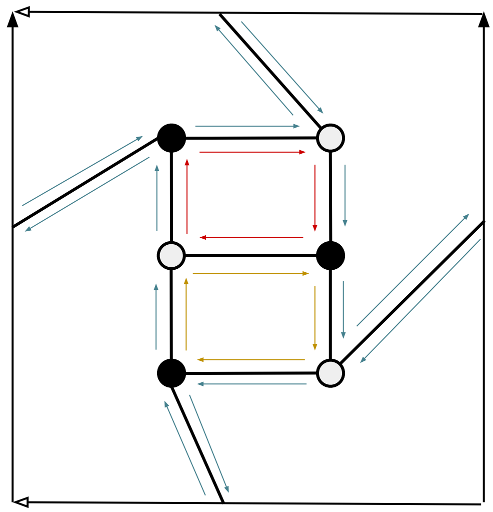

Figure 7: The faces of $K_{3,3}$ embedded in the torus are outlined in three colors: red, yellow, and blue. These are obtained by following edges around until each edge is used twice, since each edge should appear on the boundaries of the faces two times.

We will not prove this theorem, but the proof is very similar to that of the original Euler's Theorem. The primary difference is in understanding what the base case looks like: rather than a tree for the base case, we will take a graph that, when you remove edges, could be embedded in a torus with $g-1$ holes. This is, in some sense, a "minimal" genus g graph. We can then induct on the number of edges in G from this base case as we did in class.

3 Open Questions

We close this set of notes with a few open and interesting questions in the universe of graph embeddings.

First, we note that it is known that every graph can be embedded in a torus having g holes, provided that g is large enough. Indeed, this is easy to see, since we could simply add a separate hole in the torus for every single edge, completely eliminating the chance of edges crossing. (This is obviously overkill, but it will do to prove the theorem.) In the next set of notes, we will see that in order to ensure a graph is planar, it is sufficient to show that the graph does not contain $K_{3,3}$- or K_5-like subgraphs (see the next set of notes for an explanation of what we mean by G-like). It is also known that there are sets of graphs we must forbid in order to be embeddable on the torus. But in general:

Question 1. *What properties of a graph ensure that it can be embedded on a g-holed torus for some specified g?*

Further, you will notice that something odd occurred around the faces of the embedding of $K_{3,3}$ shown in Figure 7. For the blue face (the big one), we ended up using some edges twice in the boundary. That means that if I were to write out the boundary of the faces, I would not have cycles in my graph G, but I would have circuits; that is, closed walks where some edges and vertices may be used more than once. This is certainly less than desirable, and it is preferable that the faces be bounded by proper cycles in G for many reasons.

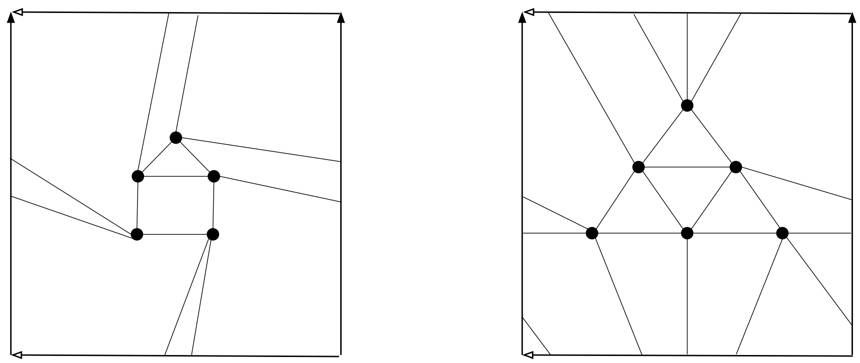

Figure 8: Two graphs embedded on the torus. The astute observer will recognize these as the complete graphs on 5 and 6 vertices, respectively, neither of which can be embedded into the plane.

Certainly we cannot ensure that faces are bounded in proper cycles of G if the graph G itself has a bridge (try drawing a planar graph with a bridge to see what we mean). Hence, we should forbid bridges if we'd like to attempt such an embedding. Moreover, we will assume that G has no cut-vertices; that is, there is no vertex whose removal would disconnect G. If we did have such a vertex, we could simply embed each portion of the graph, and then paste the embeddings together at the cut-vertex itself (see, for example, your homework, in which you prove that a planar graph can be embedded by blocks and then glued together).

Hence, we shall use the following restricted class of graphs. A graph G is said to be *2-connected* if G is connected, G contains no cut-vertices, and G contains no bridges. (Actually, written this way, this is overkill, since a bridge automatically forces a cut-vertex, but no matter). In fact the definition is written in this way because in order to disconnect G, you have to remove at least 2 things. Removing one vertex or one edge will never be sufficient to disconnect the graph. So you can see how we might have a similar definition for k-connected, for some other choice of k; a graph is *k-connected* if you have to remove k vertices to disconnect it. (We don't need this, but I might as well tell you.)

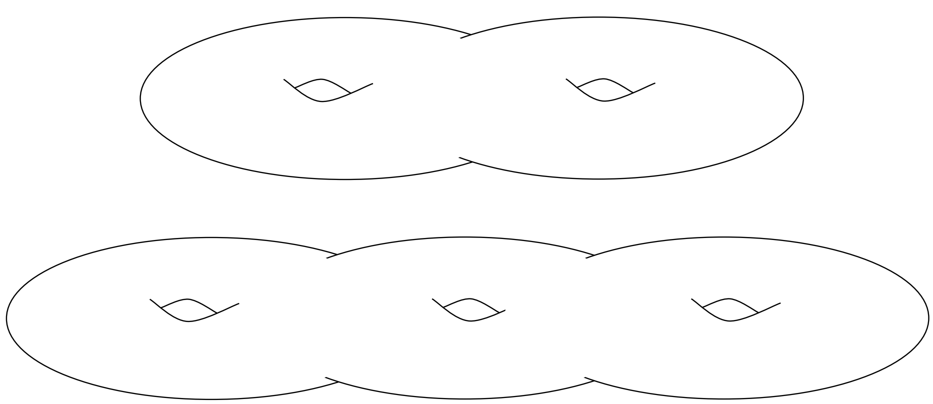

Figure 9: The two-holed torus (genus 2) and three-holed torus (genus 3). You can imagine how this will continue.

Once we have restricted into the class of 2-connected graphs, the question of embedding a graph such that the boundaries of every region form proper cycles of G now has meaning. This is a long-standing and important conjecture in graph theory:

Conjecture 1 (Strong Embedding Conjecture)**.** *Let G be a 2-connected graph. Then there exists an embedding of G into a g-holed torus (for some g) such that the boundary of every face in the embedding is a cycle in G.*

You can see that the embedding we have given of $K_{3,3}$ does not satisfy this requirement, but the embeddings given in Figure 8 do (check it out!). You should be able to construct an embedding of $K_{3,3}$ into the torus that does satisfy this requirement (try it).

Related to the Strong Embedding Conjecture is the following conjecture.

Conjecture 2 (Cycle Double Cover Conjecture)**.** *Let G be a 2-connected graph. Then there exists a (multi)-set of cycles in G such that every edge of G is used in the set of cycles exactly twice.*

That the SEC implies the CDCC is clear; any strong embedding of a graph yields a list of cycles, namely those that bound the regions. Those cycles then use every edge exactly twice, since each edge is on the boundary of exactly two regions. The opposite implication is not true; the CDCC is decidedly weaker than the SEC.

There are many other open conjectures related to the SEC and the CDCC, and I encourage you to read about these fascinating open problems! If you'd like more information about either of these, or their related problems, I'd love to share it with you. Just let me know!