Math 228: Kuratowski's Theorem

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1 Introduction

In this set of notes, we seek to prove Kuratowski's Theorem:

Theorem 1 (Kuratowski's Theorem)**.** *Let G be a graph. Then G is nonplanar if and only if G contains a subgraph that is a subdivision of either $K_{3,3}$ or K_5.*

In order to prove this theorem, let's first walk through some the definitions here, and verify that both $K_{3,3}$ and K_5 are nonplanar.

First, let's consider $K_{3,3}$. As was seen in the previous set of notes regarding graph embeddings, $K_{3,3}$ can be embedded on the torus. It was asserted in those notes that $K_{3,3}$ is not planar, but it was not proved. Hence, let us prove that assertion here.

Proposition 1. $K_{3,3}$ is not planar.

Proof. Let us prove by contradiction. Suppose, to the contrary, that $K_{3,3}$ is planar. Then there is a plane embedding of $K_{3,3}$ satisfying $v - e + f = 2$, Euler's formula. Note that here, $v = 6$ and $e = 9$.

Moreover, since $K_{3,3}$ is bipartite, it contains no 3-cycles (since it contains no odd cycles at all). So each face of the embedding must be bounded by at least 4 edges from $K_{3,3}$. Moreover, each edge is counted twice among the boundaries for faces. Hence, we must have $f \leq 2e/4 = e/2 = 4.5$.

Now, plugging this data in to Euler's formula, we obtain

$$2 = v - e + f \leq 6 - 9 + 4.5 = 1.5,$$

which is clearly false. Hence, it cannot be that $K_{3,3}$ is planar. □

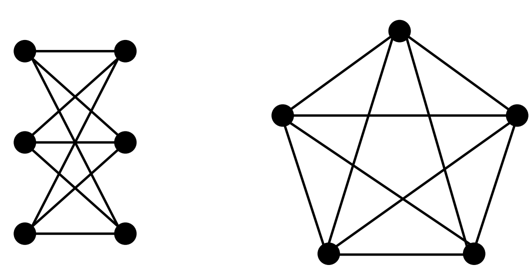

Figure 1: The two nonplanar graphs $K_{3,3}$ and K_5 discussed in the introduction, and crucial to Kuratowski's Theorem.

Now we turn to K_5. To prove that K_5 is nonplanar, we appeal to a Problem 1 from Homework 9.

Proposition 2. K_5 is not planar.

Proof. From Problem 1 in Homework 9, we have that a planar graph must satisfy $e \leq 3v - 6$. Note that for K_5, $e = 10$ and $v = 5$. Since $10 \not\leq 9$, it must be that K_5 is not planar. $\square$

2 Subdivisions and Subgraphs

Good, so we have two graphs that are not planar (shown in Figure 1). It is also straightforward to notice that if we took one of the edges from one of these graphs, and replaced it with a path of length 2 (essentially, stick another vertex in the middle of the edge), then the graph should still be nonplanar (see Figure 2). Indeed, adding this extra vertex in the middle of the edge doesn't change the fundamental *shape* of the structure, which is what makes it nonplanar to begin with.

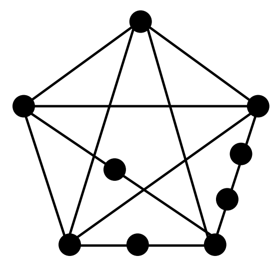

Figure 2: The graph K_5 after subdividing some edges. Notice that the shape of the structure is still unchanged, even with extra vertices having been included along some edges, and hence the structure is still nonplanar.

Let us formally define this as a *subdivision*, as follows. A graph H is said to be a *subdivision* of a graph G if H can be obtained from G by successively deleting an edge in G, and replacing that edge with a length 2 path (whose central vertex was not originally part of G). An edge that has been removed and replaced with a length 2 path is said to be *subdivided* in H. Fundamentally, we can just think of taking the edge, and dividing it into two pieces to form two different edges. So the subdivision of K_5 shown in Figure 2 is obtained by making 4 subdivisions, one along the bottom edge, one along the edge in the middle of the star, and two along the rightmost exterior edge. To formalize what we have discussed:

Lemma 1. *Let G be a graph. Then G is planar if and only if every subdivision of G is planar.*

That is to say, the act of subdividing a graph does not change the planarity of the graph at all, since the fundamental shape (the topological shape) has not changed.

We note also here that quite trivially, if we have a planar graph, and we take a subgraph, it too must be planar. Indeed, we can simply take the original graph, embed it in the plane, and then remove any edges or vertices not present in the subgraph to produce a plane drawing of the desired subgraph. This is, certainly, a very trivial property, but as it plays a fundamental role in Kuratowski's Theorem, I feel compelled to give it an entire lemma.

Lemma 2. *Let G be a planar graph. Then every subgraph of G is also planar.*

We are now set up to begin dissecting both the statement and the proof of Kuratowski's Theorem.

3 Kuratowski's Theorem: Setup

We begin this section just by restating the theorem from the beginning of the introduction, to remind ourselves what we are doing here.

Theorem 1 (Kuratowski's Theorem)**.** *Let G be a graph. Then G is nonplanar if and only if G contains a subgraph that is a subdivision of either $K_{3,3}$ or K_5.*

Note that one direction here is made trivial by the lemmas presented in the previous section. Indeed, if G contains a nonplanar subgraph, then Lemma 2 immediately implies that G is nonplanar. But by the discussion in the introduction, we also know that $K_{3,3}$ and K_5 are nonplanar, so if G contains either of these, it should be nonplanar. Allowing for subdivisions allows us to colloquially phrase Kuratowski's Theorem as follows:

Theorem 1 (Kuratowski's Theorem, layman's terms)**.** *We know two nonplanar graphs, they are $K_{3,3}$ and K_5. So of course any graph containing those is not planar. In fact, any graph containing something that has the same basic shape as those is nonplanar (that's the subdivision thing). And not only that, but every nonplanar graph has one of these two bad shapes inside it as a subgraph. That's really the only way to be nonplanar.*

This is the crux of the theorem: the only way to be nonplanar is to have one of these two known bad shapes as a subgraph.

So, we begin the proof. Before we begin, let me just remind you of a few definitions that will come in handy.

A *cut-vertex* in a graph G is a vertex v such that $G\backslash\{v\}$ has more components than G itself. That is, it's a vertex whose removal disconnects some part of the graph that used to be connected.

A *block* in a graph G is a subgraph B of G such that B has no cut vertices, but if we add any other vertices to B, it does have cut vertices (that is, it is a maximal subgraph in G having no cut vertices). In this way we can view any graph G as being built of blocks, that are simply pasted together at the cut-points.

A graph is called *2-connected* if it is connected and has no cut-vertices. We can think of 2-connected as "if you want to disconnect it, you'll have to take away 2 things." (In this way, we can generalize to "k-connected" by just replacing the number 2 with the number k in the above quotated phrase, and it will be correct.)

We have one more (nontrivial) lemma before we can begin the proof of the theorem in earnest.

Lemma 3. *Let G be a 2-connected graph, and u, v vertices of G. Then there exists a cycle in G that includes both u and v.*

Proof. We will prove this by induction on the distance between u and v.

First, note that the smallest distance is 1, which can be achieved only if u is adjacent to v. Suppose this is the case. Note that u cannot have degree 1, since otherwise, it must be that v is a cut vertex (see Figure 3). Hence, u must have another neighbor in G, say w.

Let us consider the graph $G\backslash\{u\}$. Notice that this graph is still connected, by the definition of 2-connectedness, and hence there exists a path in $G\backslash\{u\}$ between w and v. Moreover, this path cannot use the vertex u, since it has been deleted from the graph.

Adding u to this path on both ends in G creates a cycle in G, that contains both u and v.

Now, let us suppose the result is known for any u, v having distance at most $d-1$. Let $u, v \in V(G)$ have distance exactly d. Let Q be a path of length d between u and v. Take w to be the point in the path Q adjacent to v. Note that $d(u, w) = d - 1$, so there exists a cycle C in G that contains both u and w. If v is a member of this cycle, then we are done, as we have a cycle that contains both u and v.

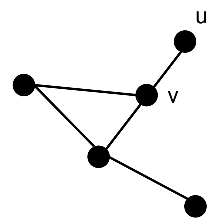

Figure 3: Notice that in the case that u is of degree 1, its neighbor v must be a cut-vertex, since deleting v would result in at least two components, one of them being just u by itself, and the others containing the rest of the vertices.

If not, then v appears outside the cycle. Moreover, since $G\backslash\{w\}$ is connected, there exists a path P between u and v that does not include the vertex w. We note that this path can contain vertices of the cycle C, just not the vertex w. Draw this as shown in Figure 4, and then create a cycle including both u and v by tracing around the exterior of the picture. □

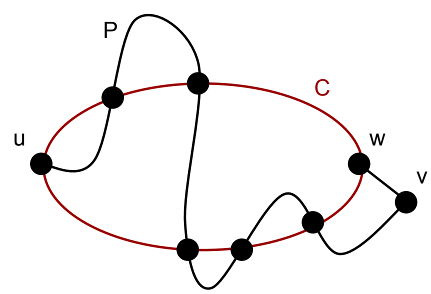

Figure 4: The structure of C and P in the case that $d(u, v) = d$. Note that the cycle C is drawn in red, and the path P is drawn in black. Both these can have more vertices that are not drawn. The intersections of P with the cycle C are drawn arbitrarily, and can occur in many different ways, this is just an example.

Now, with all this setup done, we are ready to begin the proof of Kuratowki's Theorem.

4 Kuratowski's Theorem: Proof

For simplicity throughout the proof, we will use lines to indicate not just edges in the graph, but paths as well. This will keep our drawings from getting too cluttered, and will still show the shapes we are interested in (since embeddability is all, really, about shapes).

Proof of Kuratowski's Theorem. We first note that the backward direction is trivial, by immediately applying Lemmas 1 and 2.

Now, let us consider the forward direction. We wish to show that any nonplanar graph contains a subdivision of $K_{3,3}$ or K_5. Let us work by contradiction. Suppose that the theorem is not true, so that there exists a nonplanar graph having neither a subdivision of $K_{3,3}$ nor a subdivision of K_5 as a subgraph.

From among all such counterexamples, let us choose the minimal counterexample G; where here we mean G to be minimal in the sense that any graph on either fewer vertices or edges satisfies the theorem.

Claim 1. G must be 2-connected.

Proof of Claim 1. Recall from Homework 9, Problem 2 that a graph is planar if and only if every block of the graph is planar. Hence, we have that since G is nonplanar, it must contain a nonplanar block. If this were a proper subgraph, this would be a smaller nonplanar graph that does not contain a subdivision of $K_{3,3}$ or K_5, contradicting the minimality of G. Hence, G must itself be a block, in which case G must be 2-connected. □

Good, so we know our graph G does not contain any cut vertices. In addition, we can rule out the presence of vertices of degree 2 in G, as follows.

Claim 2. G does not contain any vertices of degree 2.

Proof of Claim 2. Suppose, to the contrary, that G contains a vertex of degree 2, say v. Let the neighbors of v be u and w. We consider two cases, according to if u is adjacent to w or not.

If u is adjacent to w, consider the graph H obtained from G by removing v. By minimality of G, it must be that H is planar. Find a plane embedding of H, and then insert the path uvw next to the edge uw as shown in Figure 5. By inserting this path into the region that has uw on its boundary, we can ensure a plane embedding of G, which is a contradiction.

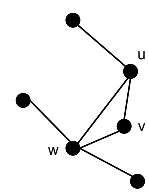

Figure 5

For the second case, if u is not adjacent to w, remove the vertex v and replace it with the edge uw to obtain a graph H. Note that this graph is smaller than G, and hence by minimality of G it must be planar. Find a plane drawing of H, and then subdivide the edge uw to produce G again. By Lemma 1, it must be the case that G is also planar, a contradiction.

As in either case we produce a contradiction, we therefore have that G cannot contain any vertex of degree 2. □

Thus, it must be that our graph G is both 2-connected, and has every vertex with degree 3 or more.

The next claim is an exercise in Homework 9.

Claim 3. G must have an edge uv such that $G\backslash\{uv\}$ is still 2-connected.

Proof of Claim 3. Problem 3 in Homework 9. □

Let $H = G\backslash\{uv\}$, the subgraph of G obtained by removing the edge (but not the vertices) u and v. We note that by the minimality of G, we must have that H is planar. Moreover, H is 2-connected, so there must exist at least one cycle in H that includes both the vertices u and v by Lemma 3.

Form a plane embedding of H in such a way that there is a cycle C satisfying the following:

- C contains both u and v
- The number of regions inside of C in the embedding is maximal among all other embeddings.
- If C' is any other cycle that contains both u and v, the number of regions inside C' in a plane embedding of H is less than (or equal to) the number of regions inside C.

That is to say, we have chosen a cycle containing u and v so that the number of regions inside the cycle is maximized among ALL cycles containing u and v, among ALL embeddings of H. Write this cycle C as $u = v_0, v_1, v_2, \ldots, v_{k-1}, v_k = v, v_{k+1}, \ldots, v_\ell, v_0$. Note that it must be the case that $k \geq 2$, since u and v are not adjacent in H.

We make the following observations.

- There is no path connecting two vertices in the set $\{v_0, v_1, \ldots, v_k\}$ that lies exterior to C.
- There is no path connecting two vertices in the set $\{v_k, v_{k+1}, \ldots, v_\ell, v_0\}$ that lies exterior to C.

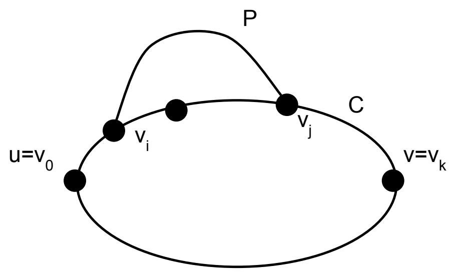

Figure 6: A path P as described in the proof of the observations, connecting two vertices between v_0 and v_k.

These two observations are proved in the same way, and hence we shall only address the first one here. Consider the graph shown in Figure 6, supposing that there is a path of this type between vertices v_i and v_j. Note then that we could construct a cycle C' including u and v as $v_0, v_1, \ldots, v_i, P, v_j, v_{j+1}, \ldots, v_\ell, v_0$ (this is obtained by just tracing around the exterior of the drawing in Figure 6). Note further that this cycle C' contains all the regions that C contains, plus at least one more. This contradictions the maximality property for C, and hence it is impossible. The exact same construction suffices to prove the other observation.

However, we cannot add the edge uv, since G is not planar, so it must be that there is a path lying exterior to C that connects some vertex in $\{v_1, v_2, \ldots, v_{k-1}\}$ to some vertex in $\{v_{k+1}, v_{k+2}, \ldots, v_\ell\}$. Say these vertices are v_i and v_j. This is shown in Figure 7.

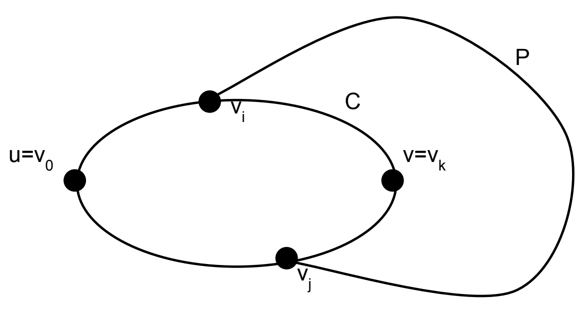

Figure 7: The necessary path P that blocks us from adding the edge uv exterior to C in the embedding of H.

We note that no vertex of P is adjacent to any other vertex of C, as we would necessarily have two vertices on the same side of v_k and hence be in the case observed above, in which we can build a cycle containing both u and v and more regions.

Now, it must be the case that we could not have placed the path P interior to C, otherwise we would do so and thus be able to include the edge uv exterior to C. Hence, there must be some impedimentary structure inside C blocking us from drawing P in the interior.

We claim, without proof (although you should be able to prove this!) that the impedimentary structure inside C must take one of the forms illustrated in Figure 8.

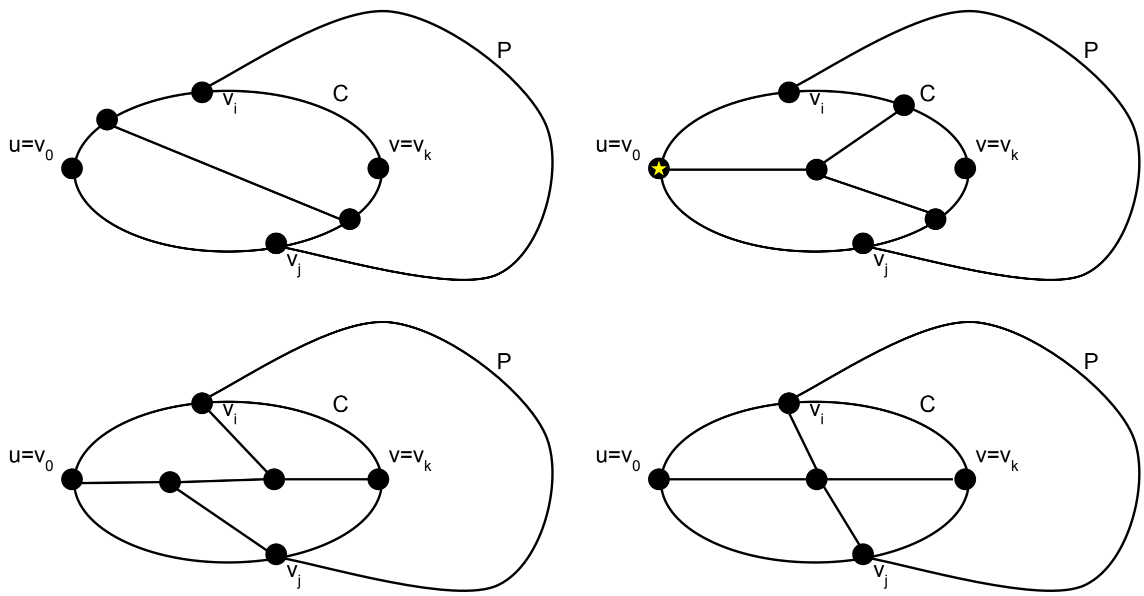

Figure 8: The possible impediments to including the edge uv in the embedding of H. Note here that all lines indicate paths, not edges. In the top right structure, we have indicated one vertex with a star. In fact, this impedimentary structure could exist in many different ways, by choosing any of v_0, v_i, v_k, v_j as the starred vertex, and setting the other two vertices in a symmetric way. All these cases have identical analysis, though, so we shall only deal with one such.

We note that in any of these four cases, when we add back in the edge uv as we have in G, we will be able to find a subdivision of $K_{3,3}$ or K_5 inside G, as shown in Figure 9. For the sake of simplicity, in that figure we have removed all vertex labels, and highlighted the desired substructures using some colors.

Therefore, we have reached a contradiction: G does contain a subgraph that is a subdivision of either $K_{3,3}$ or K_5. Hence, our initial assumption must be untrue: there is no nonplanar graph that does not contain a subdivision of $K_{3,3}$ or K_5 as a subgraph. Therefore, the theorem holds.

□

5 Generalizations and the Topological Minor Theorem

Kuratowski's Theorem is an excellent, useful way of determining if a graph is planar or not. But as we saw in previous notes, if a graph is not planar, then we'd like to try to embed it in a torus with as few holes as possible. So there is a natural question: Is there a generalization of Kuratowski's Theorem that will tell us when a graph embeds into a torus with g holes? The answer, through a very beautiful and

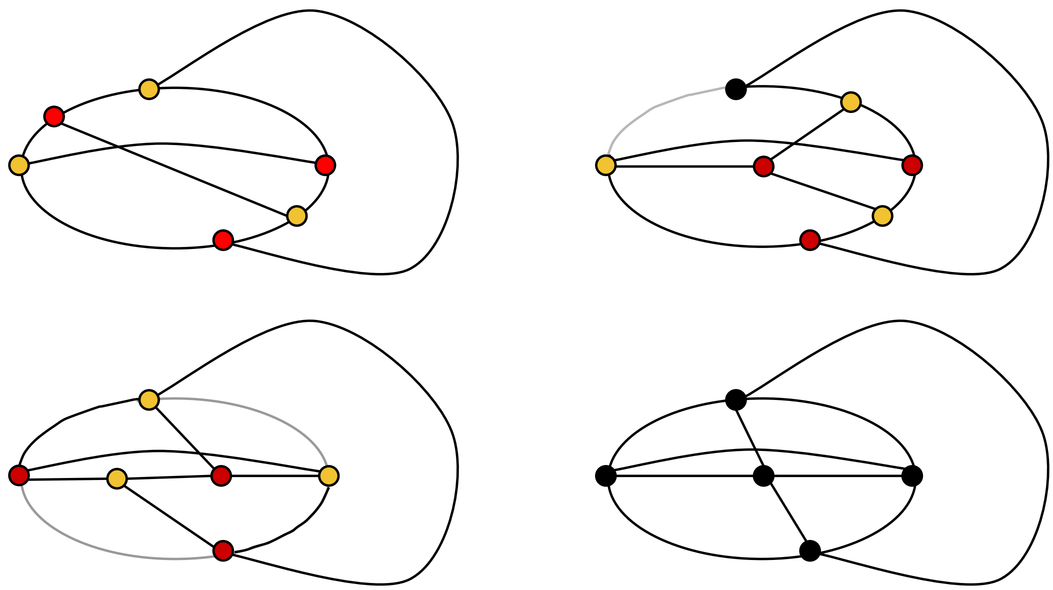

Figure 9: Here we find the subgraphs that are subdivisions of $K_{3,3}$ or K_5 in each of the impedimentary structures shown in Figure 8. For the top two graphs, as well as the bottom left, we have a subdivision of $K_{3,3}$. The two partite sets are shown in yellow and red, and any paths used to construct the subgraph are black. Any edges that are greyed out are not needed in the construction of the subgraph. In the bottom right graph, careful observation reveals that the graph itself is K_5 (well, a subdivision of K_5, at least, since each edge actually represents a path in G).

difficult piece of mathematics, is yes:

Theorem 2 (Topological Minor Theorem)**.** *For any $g \geq 0$, there exists a finite list of graphs $\mathcal{G}_g$ such that a graph G embeds into a torus with g holes if and only if it contains no subgraph that is a topological minor of a graph in $\mathcal{G}_g$.*

Immediately, your hand should go up, and you should say "what's a topological minor!?" The basic understanding you should take from that phrase "topological minor" is that the subgraph has the same basic shape as a graph from $\mathcal{G}_g$, just as in Kuratowski's Theorem we take subdivisions as representing the same basic shape as $K_{3,3}$ or K_5. The full definition[1] of topological minor is below in the footnote.

So, that sounds good, we can get finite lists of subgraph shapes to exclude for any kind of torus! We usually call the lists $\mathcal{G}_g$ the set of *forbidden minors* for the g-holed torus. The first followup question you should have is: what graphs are in $\mathcal{G}_g$?

Unfortunately, we don't know the answer for *any* choice of g except $g = 0$ (which is Kuratowski's Theorem itself). We do know a few things, and the things we know are not very promising.

Theorem 3. *The set of topological minors for the 1-holed torus has size at least 16000.*

Suffice it to say that 16000 is a pretty unreasonable number to expect to be working with in order to determine if a graph is or is not toroidal (i.e., has an embedding on the torus). We can narrow this down, actually, if we are picky about which minors we're interested in, but that still doesn't give us much direction in terms of the practical aspects of determining if we have a graph that can be embedded on the torus.

We do have a complete, not too gigantic list of forbidden minors for at least one other surface, and that is the projective plane. The projective plane is nonorientable, like the Möbius strip. The Möbius strip is like a circle with a twist; you can sort of think of the projective plane as like a sphere with a twist, so that when you start walking in one direction, you'll return to where you came from backwards. For this surface, we know the forbidden minors.

Theorem 4. *The projective plane has a list of forbidden minors containing only 35 graphs.*

The question of how to determine the forbidden minors for a given surface and how to prove they are correct is still unsolved. As we have seen, our proof of Kuratowski's Theorem was highly reliant on specific structure of the forbidden minors themselves. It seems unlikely that such a proof could be replicated on the torus, where the minor set is huge. It is likely these proofs will end up being performed by computers, if and when we have a candidate set for forbidden minors.

[1] To define topological minor, we must first define contraction along an edge. Let G be a graph, and let uv be an edge of G. The *contraction of G along uv*, often denoted by $G/\{uv\}$ is the graph obtained by deleting the vertices u and v, and replacing them with a single vertex w, where w is adjacent to every neighbor of u and v (if they have a common neighbor, this will result in a multiple edge out of w). That is, think of squeezing u and v together into a single vertex, with out affecting any of their neighbors. If you think about embedding a graph, certainly if you can embed a graph G into some surface, the embedding will not really change if we perform contraction, so you should be able to perform any contraction you wish and still keep the existing embedding.

A *topological minor* of a graph G is any graph H that can be transformed into G by a series of contractions, edge deletions, and/or removal of isolated vertices. So the original H should have the structure essentially of G inside of it, but we need to smush things around a little to get it to look *exactly* like H. This is the idea of the Minor Theorem: G morally contains a subgraph that "looks like" something from $\mathcal{G}_g$, even if it's not exactly that graph.