Math 127 Homework

Mary Radcliffe

Due 7 March 2019

Complete the following problems. Fully justify each response. You need only turn in those problems marked with a (*).

Note: the first two problems are repurposed from HW5. You may have already completed these problems.

1. Given an integer d, define the set $A_d = \{n \in \mathbb{N} \mid n \text{ is divisible by } d\}$.
   (a) Suppose p, q are distinct primes. Show that $A_p \cap A_q = A_{pq}$.
   (b) Suppose a, b are integers with $a|b$. Prove that $A_b \subseteq A_a$.
2. (*) Let X, Y be sets in a universe Ω.
   (a) Prove that $X \subseteq Y$ if and only if $X \cap Y = X$.
   (b) Prove that $X \subseteq Y$ if and only if $X \cup Y = Y$.
3. Prove Theorem 3 (De Morgan's Laws, v2) from the Set Theory Notes.
4. Prove the following De Morgan's Law variant for relative complements:

   Let $B, A_1, A_2, \ldots, A_n$ be sets. Then
   - $B \backslash \left(\bigcup_{k=1}^{n} A_k \right) = \bigcap_{k=1}^{n} (B \backslash A_k)$, and
   - $B \backslash \left(\bigcap_{k=1}^{n} A_k \right) = \bigcup_{k=1}^{n} (B \backslash A_k)$
5. (*) Sets A and B are called *disjoint* if $A \cap B = \emptyset$. A collection of sets $A_1, A_2, \ldots, A_n$ are called *pairwise disjoint* if $A_i \cap A_j = \emptyset$ for all $i \neq j$.

   Prove that for any sets A, B, $A \cup B = (A \backslash B) \cup (B \backslash A) \cup (A \cap B)$, and that the sets $A \backslash B, B \backslash A, A \cap B$ are pairwise disjoint.
6. Prove the following set identities.
   (a) $A \backslash B = A \cap B^c$
   (b) $A \cup (A \cap B) = A$
   (c) $A \cap (A \cup B) = A$
   (d) $(B \cup C) \backslash A = (B \backslash A) \cup (C \backslash A)$
   (e) $(B \cap C) \backslash A = (B \backslash A) \cap (C \backslash A)$
7. (*) Use the Well-Ordering Principle to prove the Archimedean Property: For all $a, b \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $na > b$.
8. For each of the following, give an example of a set in $\mathbb{R}$ that meets the description.

(a) Has a supremum, but does not contain a maximum.

(b) Contains both a maximum and a minimum.

(c) Has neither supremum nor infimum.

9. Use the definition of upper and lower bounds to explain why every number is both a lower and upper bound for $\emptyset$. As a result, we sometimes say that $\sup(\emptyset) = -\infty$ and $\inf(\emptyset) = \infty$; explain this convention.

10. Use the Well-Ordering Principle to prove that $\forall n \in \mathbb{N}$, $n! \leq n^n$.

11. Prove Parts 1, 4 of Proposition 1 in the Functions Notes.

12. (*) Prove Proposition 2 in the Functions Notes.

13. Which of the following function specifications are well-defined? If one is not well-defined, determine a modification to the specification that would rectify the issue.

    (a) $g : \mathbb{Q} \to \mathbb{Q}$ defined by $(x+1)g(x) = 2$ for all x.

    (b) $f : \mathbb{Q} \to \mathbb{R}$ defined by $(x+\pi)f(x) = 1$ for all x.

    (c) $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \sqrt{x}$.

14. Let $f, g, h, \ell : \mathbb{R} \to \mathbb{R}$ be functions with the following specifications:

$$f(x) = x+2; \quad g(x) = x^2; \quad h(x) = \frac{1}{x^2+1}; \quad \ell(x) = -x.$$

    Write a specification, via a single equation, for each of the following:

    (a) $f \circ g$.

    (b) $g \circ f$.

    (c) $f \circ (g \circ (h \circ \ell))$.

    (d) $(f \circ g) \circ (h \circ \ell)$.

15. (*) For each of the following functions, determine if it is injective, surjective, both, or neither. Prove that your answers are correct.

    (a) $f : \mathbb{Z} \to \mathbb{N} \cup \{0\}$, $f(x) = x^2$.

    (b) $g : \mathbb{N} \to \mathbb{Z}$, $g(x) = x^2$.

    (c) $h : \mathbb{R} \to \mathbb{Z}$, $h(x) = \lfloor x \rfloor$

    (d) $f : \mathbb{N} \to \mathbb{Z}$, $f(x) = \begin{cases} \frac{x}{2} & x \text{ is even} \\ -\frac{x-1}{2} & x \text{ is odd} \end{cases}$

16. Prove that if $f : X \to Y, g : Y \to Z$ are functions on sets X, Y, Z, and $g \circ f$ is injective, then f is injective.

17. (*) Prove that if $f : X \to Y, g : Y \to Z$ are functions on sets X, Y, Z, and $g \circ f$ is surjective, then g is surjective.