Math 127 Homework

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Complete the following problems. Fully justify each response. You need only turn in those problems marked with a (*).

1. Given $n \geq 0$, recursively define $n!$ by

$$0! = 1, \quad , n! = n(n-1)! \text{ for } n > 0.$$

   Prove that for every $n \geq 4$, $2^n < n!$.

2. Prove that for any $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} k!k = (n+1)! - 1.$$

3. (*) Prove that for any $n \in \mathbb{N}$,

$$\prod_{i=1}^{n} (2i-1) = \frac{(2n)!}{2^n n!}.$$

4. Let $a_0, a_1, a_2, a_3, \ldots$ be a sequence of real numbers, defined recursively by

$$a_0 = 0; \quad a_1 = 1; \quad a_n = 7a_{n-1} - 12a_{n-2} \text{ for } n \geq 2.$$

   Prove that for all $n \geq 0$, we have $a_n = 4^n - 3^n$.

5. (*) Prove, using induction, that for any odd $n \in \mathbb{N}$, we have $n^2 - 1$ is divisible by 8. How does the fact that n is odd change the structure of the proof?

6. Suppose we play a game where there are 2 players. The game is as follows: first, make two nonempty piles of pennies. On each players turn, they may remove as many pennies as they like from one of the piles (but not 0). The player who removes the last penny wins the game.

   Prove that if the two piles initially contain the exact same number of pennies, then the second player can always win the game. Prove that if the two piles initially contain different numbers of pennies, then the first player can always win the game.

7. (*) Suppose you have a $2 \times n$ playing board, and a supply of 2×1 dominoes. Let D_n denote the number of ways to cover the board with dominoes. For example, it is plain to see that $D_1 = 1$ and $D_2 = 2$.

   Determine an expression for the value of D_n. Prove that your expression is correct.

8. (*) Show that any $n \in \mathbb{N}$ can be written in the form $n = d_1 1! + d_2 2! + \cdots + d_r r!$, where each d_i satisfies $0 \leq d_i \leq i$.

9. Given an integer d, define the set $A_d = \{n \in \mathbb{N} \mid n \text{ is divisible by } d\}$.

   (a) Suppose p, q are distinct primes. Show that $A_p \cap A_q = A_{pq}$.

(b) Suppose a, b are integers with $a|b$. Prove that $A_b \subseteq A_a$.

10. (*) Let X, Y be sets in a universe Ω.

    (a) Prove that $X \subseteq Y$ if and only if $X \cap Y = X$.

    (b) Prove that $X \subseteq Y$ if and only if $X \cup Y = Y$.