Math 127 Homework

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Complete the following problems. Fully justify each response. You need only turn in those problems marked with a (*).

1. Let x be a real number, and write a decimal expansion of x in the form $x = n.d_1d_2d_3d_4\dots$, where n is an integer, and $d_1, d_2, d_3, \dots$ are digits of the decimal expansion. Let $x_1, x_2, x_3, \dots$ be a sequence as defined on page 5 of the notes, so $x_1 = n$, $x_2 = n.d_1$, $x_3 = n.d_1d_2$, etc. Prove that, using the definition given, x is the limit of the sequence.

2. (*) Use the Peano Axioms to prove that multiplication in the natural numbers is commutative.

3. Use the Peano Axioms to prove that addition in the natural numbers is associative.

4. (*) Use mathematical induction to prove that for any $n \in \mathbb{N}$, $4^n + 6n - 1$ is divisible by 9.

5. (*) Use mathematical induction to prove that for any $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$.

6. (*) Prove Theorem 3 in the Induction notes.

7. (*) Here is a "proof" by induction of something we know to be false. Explain why it is broken.

   Proposition 1. For any $n \in \mathbb{N}$, $\displaystyle\sum_{i=0}^{n} 2^i = 2^{n+1}$.

   Proof. We work by induction on n. Suppose that for some $n \in \mathbb{N}$, the result is true, so that $\displaystyle\sum_{i=0}^{n} 2^i = 2^{n+1}$. Consider the case of $n+1$. We have

$$\begin{aligned}\sum_{i=0}^{n+1} 2^i &= \sum_{i=0}^{n} 2^i + 2^{n+1} \\ &= 2^{n+1} + 2^{n+1} \quad \text{(by the inductive hypothesis)} \\ &= 2 \cdot 2^{n+1} = 2^{n+2}.\end{aligned}$$

   Therefore, if the result is true for n, it is also true for $n+1$. Hence, the result holds for all $n \in \mathbb{N}$. □

8. Here is a "proof" by induction of something we know to be false. Explain why it is broken.

   Proposition 2. If you have finitely many horses, then every horse is the same color.

Proof. Number the horses $h_1, h_2, h_3, \ldots$. We work by induction on n to show that $h_1, h_2, \ldots, h_n$ are all the same color for any fixed n. Since there are finitely many horses, this implies that all horses are the same color.

Now, for the base case, suppose $n = 1$. Then we have only one horse, and hence all the horses we have are the same color.

Now, suppose the result is true if we have n horses. Consider a set $h_1, h_2, \ldots, h_n, h_{n+1}$ of $n + 1$ horses. Note that by the induction hypothesis, the set $h_1, h_2, \ldots, h_n$ of horses are all the same color, say brown. Also by the induction hypothesis, since $h_2, \ldots, h_n, h_{n+1}$ is a set of only n horses, they are also all the same color. Since h_2 is in this set, and we know already that h_2 is brown, we must have that all these horses are also brown. Hence, all of $h_1, h_2, \ldots, h_n, h_{n+1}$ are the same color.

Therefore, by mathematical induction, if you have finitely many horses, then every horse is the same color. □